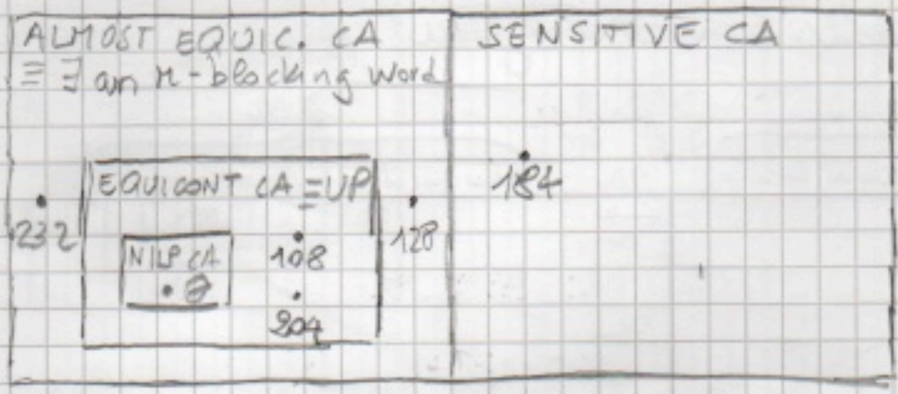
CLASSIFICATION OF CA WRT STABILITY / INSTABILITY



PROPOSITION Let $\langle A, r, f \rangle$ be a CA
It is DECIDABLE to establish if a given word $u \in A^+$ is r-blocking

PROPOSITION
The problem of establish if a CA admits an r-blocking word is UNDECIDABLE (it is rec. enumerable, or, in other terms, semi-decidable)

A semi-algorithm:

```
n = r
while (true)
    for each u ∈ A^n
        if u is r-blocking
            return true
    n++
```

COROLLARY
It is UNDECIDABLE to establish whether a CA is almost equicontinuous or sensitive

PROPOSITION

It is UNDECIDABLE to establish whether a CA is nilpotent

proof we will see later..

CORollARY

It is UNDECIDABLE to establish whether a CA is equicontinuous (or, equivalently, ultimately periodic)

proof (sketch)

For a sake of argument, let us assume the UP is decidable. Let $\langle A, r, f \rangle$ be any CA and let F its global rule. Since UP is decidable, we are able to establish whether F falls into one of the following cases

1) F is not UP. Then, F is not nilpotent
2) F is UP. Then, $\exists\, p>0, q \in \mathbb{N}$ such that $F^q = F^{q+p}$. By using spatially periodic configurations p and q are computable.

2a) $p \neq 1 \Rightarrow F$ is NOT NILPOTENT

2b) $p = 1 \Rightarrow$ 2b.1) if ${}^\infty q^\infty$ is the unique x s.t. $F(x) = x$ for some $q \in A$ $\Rightarrow F$ is NILPOTENT

2b.2) Otherwise, F is not NILPOTENT

CA NILPOTENCY

$\exists\, zero \in A^{\mathbb{Z}}$ t.c. $\exists t \in \mathbb{N}\ \forall x \in A^{\mathbb{Z}}\ F^t(x) = zero$

REMARK $\sigma(zero) = zero$ and $F(zero) = zero$

- $zero = {}^\infty q^\infty$ for some $q \in A$ (i.e., $\sigma(zero) = zero$)

Indeed, if $x \in A^{\mathbb{Z}}$ is any configuration we have $\sigma(zero) = \sigma(F^t(x)) = F^t(\sigma(x)) = zero$. Moreover $F(zero) = F(F^t(x)) = F^t(F(x)) = zero$.

A SUFFICIENT CONDITION FOR CA SENSITIVITY

REMARK Let F be a CA of local rule $f: A^{2r+1} \to A$.
For any $t \in \mathbb{N}$, let $f^{(t)}: A^{2rt+1} \to A$ be the local rule of the CA F^t.
If f is leftmost/rightmost permutive then $f^{(t)}$ is so.

PROPOSITION Let F be a CA with L/(R) permutive local rule f. Then, F is sensitive to the initial conditions.

proof

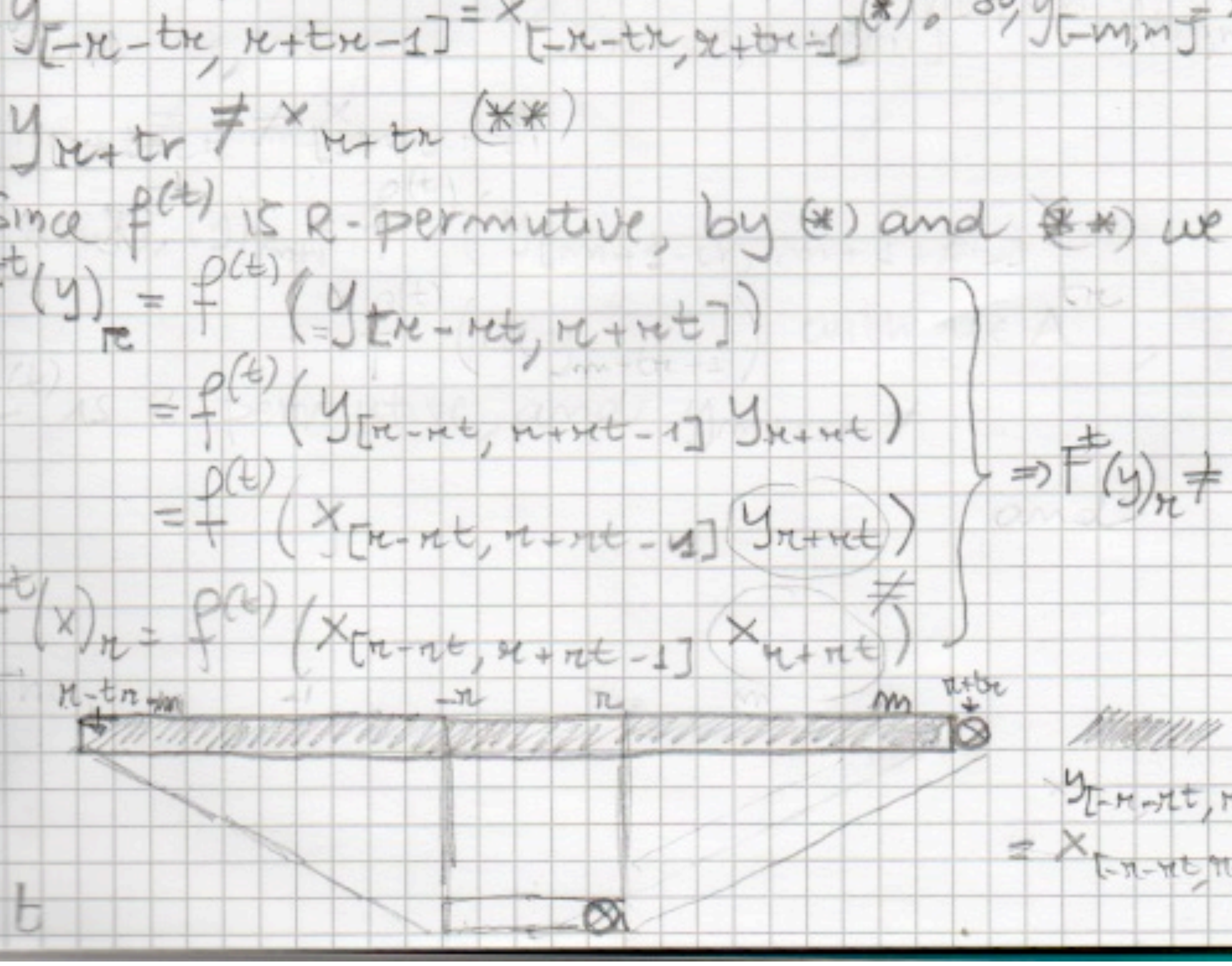
We are going to show that $\exists\, \varepsilon = r$ s.t. $\forall x \in A^{\mathbb{Z}}\ \forall m \in \mathbb{N}$
$\exists\, y \in A^{\mathbb{Z}}$ $(y \neq x)$ with $y_{[-m,m]} = x_{[-m,m]} \wedge \exists\, t \in \mathbb{N}$
$F^t(y)_{[-r,r]} \neq F^t(x)_{[-r,r]}$.

Choose arbitrarily $x \in A^{\mathbb{Z}}$ and $m \in \mathbb{N}$.
Build $y \in A^{\mathbb{Z}}$ as follows. Let $t \in \mathbb{N}$ such that
$[-m,m] \subsetneq [r - tr,\ r + tr]$ and set

- $y_{[-r-tr,\ r+tr-1]} = x_{[-r-tr,\ r+tr-1]}$ (*). So, $y_{[-m,m]} = x_{[-m,m]}$
- $y_{r+tr} \neq x_{r+tr}$ (**)

Since $f^{(t)}$ is R-permutive, by (*) and (**) we get

$$\left.\begin{aligned} F^t(y)_r &= f^{(t)}\left(y_{[r-rt,\ r+rt]}\right) \\ &= f^{(t)}\left(y_{[r-rt,\ r+rt-1]}\ y_{r+rt}\right) \\ &= f^{(t)}\left(x_{[r-rt,\ r+rt-1]}\ y_{r+rt}\right) \\ & \qquad\qquad\qquad\qquad\quad \neq \\ F^t(x)_r &= f^{(t)}\left(x_{[r-rt,\ r+rt-1]}\ x_{r+rt}\right) \end{aligned}\right\} \Rightarrow F^t(y)_r \neq F^t(x)_r$$



$y_{[-r-rt,\ r+rt-1]} = x_{[-r-rt,\ r+rt-1]}$