EXAMPLES

We have already seen that f with

$m_f = 106$

$m_f = 170$ (the local rule of the CA σ)

are rightmost - permutive.

Therefore, the corresponding CA F are sensitive

Consider now the CA with $m_f = 90$

000	0
001	1
010	0
011	1
100	1
101	0
110	1
111	0

$$f(x_{-1}, x_0, x_1) = (x_{-1} + x_1) \bmod 2$$

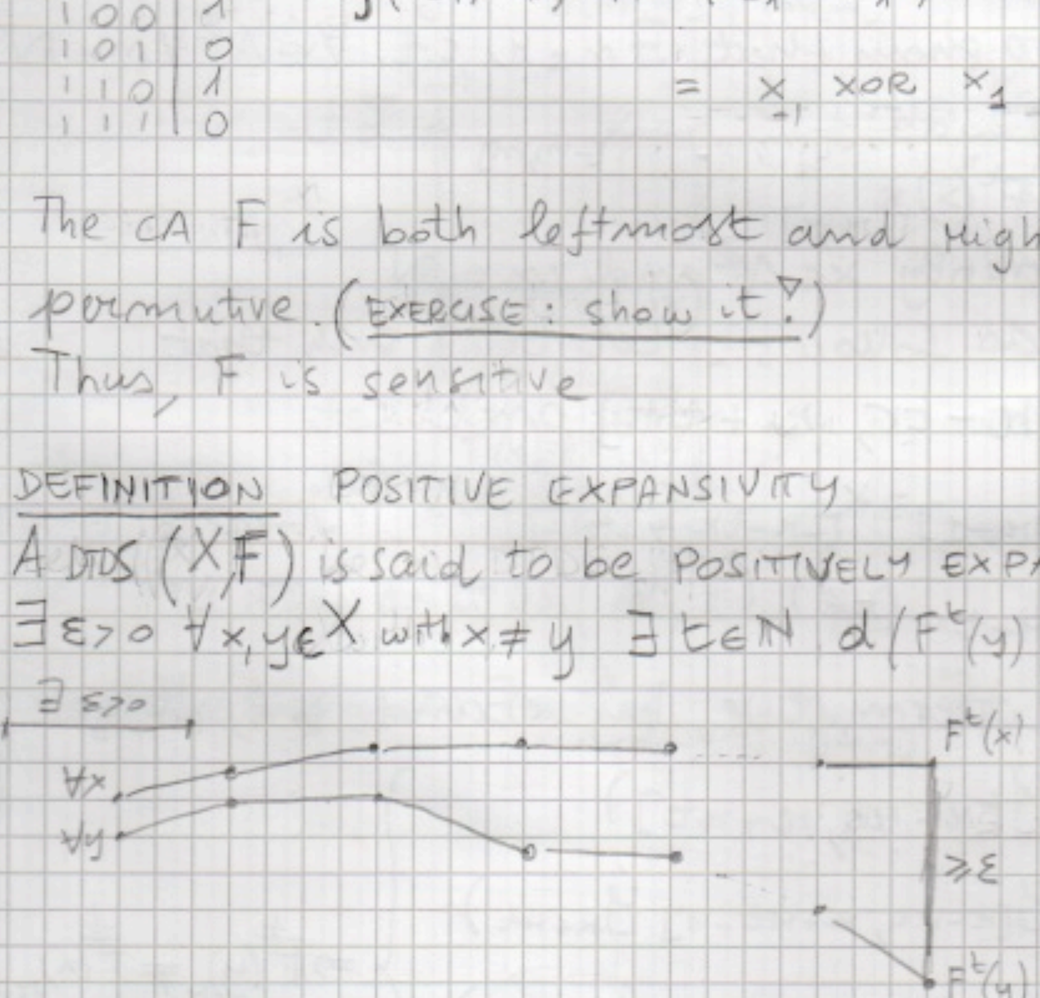
$$= x_{-1} \text{ xor } x_1$$

The CA F is both leftmost and rightmost permutive. (EXERCISE: show it!)

Thus, F is sensitive

DEFINITION POSITIVE EXPANSIVITY

A DTDS (X, F) is said to be POSITIVELY EXPANSIVE if
$\exists \varepsilon > 0\ \forall x, y \in X$ with $x \neq y\ \exists t \in \mathbb{N}\ d(F^t(y), F^t(x)) \geq \varepsilon$



In $A^{\mathbb{Z}}$

$\exists n \in \mathbb{N}\ \forall x, y \in A^{\mathbb{Z}}$ with $x \neq y\ \exists t \in \mathbb{N}\quad F^t(y)_{[-n,n]} \neq F^t(x)_{[-n,n]}$

It is easy to show that (<u>do it as exercise</u>)

PROPOSITION

Let (X, F) be a DTDS with $|X| = \infty$

If (X, F) is positively expansive then it is sensitive

<u>REMARK</u>

If a DTDS (X, F) has $|X| < \infty$ then

1) it is ULTIMATELY PERIODIC (and then equicontinuous)

2) it is EXPANSIVE

proof of 2)

Set $\varepsilon = \min\{d(x,y) \mid x, y \in X \text{ with } x \neq y\}$

Such an ε exist and $\varepsilon > 0$

Moreover $\forall x, y \in X \ d(y,x) \geq \varepsilon$

The fact that (X, F) is expansive does not mean that F is unstable.

As to CA, the following result holds

PROPOSITION

Let $(A^{\mathbb{Z}}, F)$ be a CA with local rule which is both leftmost and rightmost permutive.

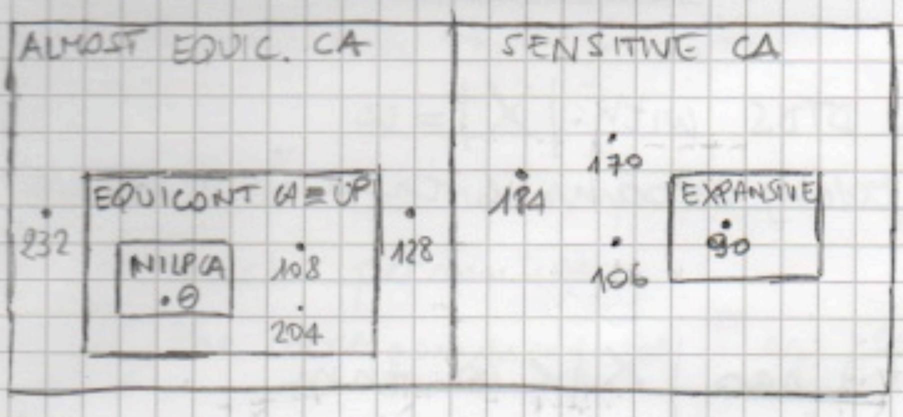
Then, F is positively expansive.

<u>proof</u> (<u>do it as exercise</u>)

<u>EXAMPLE</u>

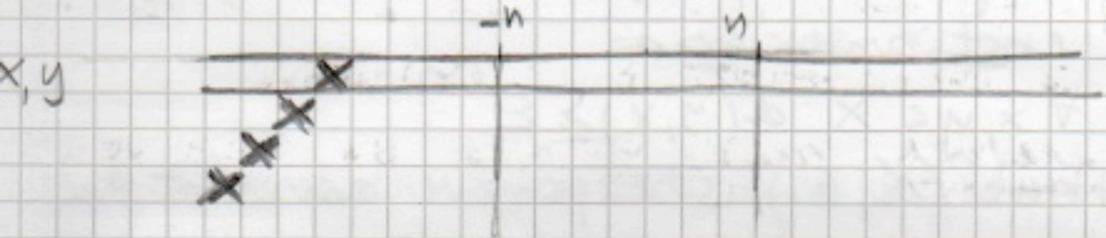
The CA F with local rule having number $n_f = 90$ both leftmost and rightmost permutive.

Thus, F is positively expansive



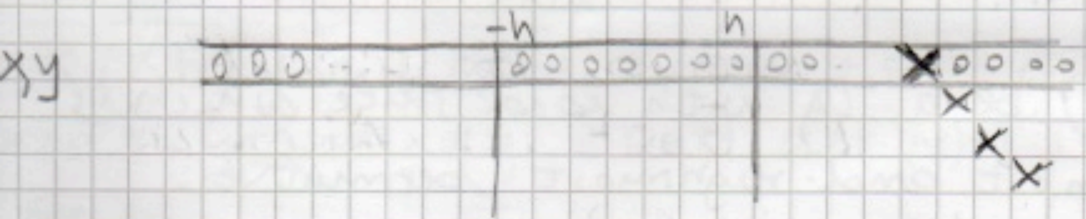
REMARK

The CA F with local rule having number $n_f = 170$, i.e., $F = \sigma$, is not positively expansive.



The same fact holds for the CA with $n_f = 106$

As to $n_f = 184$ (traffic rule)



Thus, F is not expansive.

Sensitivity is not enough to label a DTDS as chaotic.
Why?

Consider the following DTDS (X, F) where
. $X = \mathbb{R}$
. $\forall x \in X,\ F(x) = 2 \cdot x$ (F is linear)

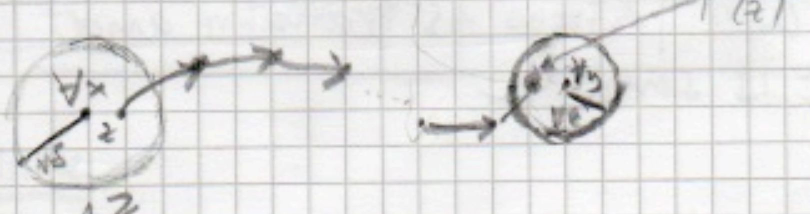
$\forall t \in \mathbb{N}\ \ F^t(x) = 2^t \cdot x$ ← This holds $\forall x \in X$!
So the dynamical evolution of initial state x is

$(x, 2\cdot x, 2^2 x, 2^3 x, \dots, 2^t \cdot x, \dots)$

Thus $\forall x, y \in X,\ d(F^t(y), F^t(x)) = |2^t y - 2^t x| = 2^t |x-y|$
It is easy to check that F is expansive
(<u>do it as exercise</u>) and then is also sensitive.

A linear system is not the system one would call chaotic... We need some further notions.

<u>DEFINITION</u> TRANSITIVITY
A DTDS (X, F) is said to be TRANSITIVE if
$\forall x \in X\ \forall \delta > 0\ \forall y \in X\ \forall \varepsilon > 0\ \exists z \in X$ with $d(z, x) < \delta$
and $\exists t \in \mathbb{N}\ d(F^t(z), y) < \varepsilon$

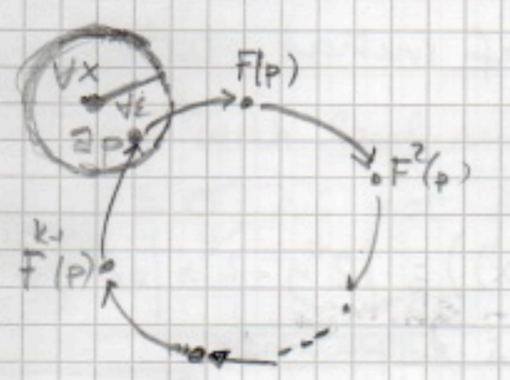


In $A^{\mathbb{Z}}$
$\forall x \in A^{\mathbb{Z}}\ \forall m \in \mathbb{N}\ \forall y \in A^{\mathbb{Z}}\ \forall m \in \mathbb{N}\ \exists z \in A^{\mathbb{Z}}$ with $z_{[-m,m]} = x_{[-m,m]}$
and $\exists t \in \mathbb{N}\ \ F^t(z)_{[-m,m]} = y_{[-m,m]}$
REMARK $(\mathbb{R}, F)$ with $\forall x \in \mathbb{R},\ F(x) = 2 \cdot x$ is NOT TRANSITIVE!
Choose $x = -2,\ y = 4$. Then $\forall t \in \mathbb{N}\ F^t(x) < 0 \wedge F^t(y) > 0 \dots$

DEFINITION DENSE PERIODIC ORBITS (DPO)

A DTDS (X, F) has DENSE PERIODIC ORBITS if $\forall x \in X \; \forall \varepsilon > 0 \; \exists$ a periodic point p s.t. $d(x,p) < \varepsilon$

→ i.e, a ultimately periodic point p with preperiod $= 0$, i.e, with orbit $(p, F(p), \ldots, F^{k-1}(p))$ for some $k > 0$



$(\mathbb{R}, F)$ with $\forall x \in \mathbb{R} \; F(x) = 2 \cdot x$ does not have DPO!
$p = 0$ is the unique periodic point of $(\mathbb{R}, F)$!

DEFINITION CHAOTIC DTDS (according to Devaney)

A DTDS (X, F) is CHAOTIC if

- $|X| = \infty$ AND
- (X, F) is sensitive AND
- (X, F) is transitive AND
- (X, F) has DPO

THEOREM

If a DTDS (X, F) with $|X| = \infty$ is transitive and has DPO, then it is sensitive

CHAOS IN CELLULAR AUTOMATA

THEOREM

If a CA $(A^{\mathbb{Z}}, F)$ is transitive then it is sensitive

If a CA $(A^{\mathbb{Z}}, F)$ is expansive then it is transitive

PROPOSITION

A CA $(A^{\mathbb{Z}}, F)$ with leftmost (or rightmost) local rule is transitive

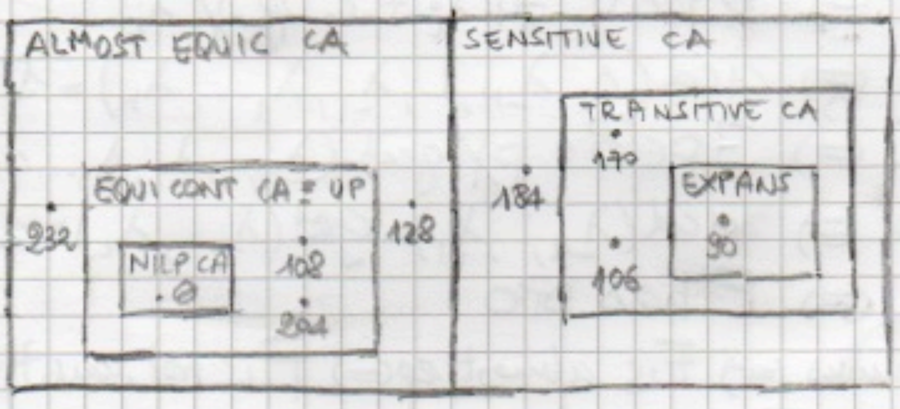
proof (do it as exercise)

PROPOSITION

A CA $(A^{\mathbb{Z}}, F)$ with leftmost (or rightmost) local rule has DPO

COROLLARY

A CA $(A^{\mathbb{Z}}, F)$ with leftmost (or rightmost) local rule is CHAOTIC



TRANSITIVITY $\equiv$ CHAOS ?

REMARK

There exist non permutive CA which are transitive or expansive. Take a L-permutive CA F and an injective CA G which is neither L- nor R-permutive. Then $G^{-1} \circ F \circ G$ is a CA which is not permutive. But it is transitive!

LINEAR CELLULAR AUTOMATA

$A = \mathbb{Z}_s = \{0, 1, \dots s-1\} = \mathbb{Z}/s\mathbb{Z} \quad \langle \mathbb{Z}_s, +, \cdot \rangle$

DEFINITION linear local rule

$f : \mathbb{Z}_s^{2r+1} \to \mathbb{Z}_s$ is linear if $\exists\ \lambda_{-r} \dots \lambda_0 \dots \lambda_r \in \mathbb{Z}_s$ such that

$\forall (x_{-r}, \dots, x_r) \in \mathbb{Z}_s^{2r+1}, \ f(x_{-r}, \dots, x_r) = \left(\sum_{i=-r}^{r} \lambda_i \cdot x_i \right) \bmod s$

DEFINITION LINEAR CA

A CA F is linear if it is defined by a linear local rule.

THEOREM Let $(\mathbb{Z}_s^{\mathbb{Z}}, F)$ a LINEAR CA defined by a liner local rule with coefficients $\lambda_{-r}, \dots, \lambda_0, \dots, \lambda_r$
Let $\mathcal{P}$ denote the set of prime factors of s.
Let us denote by $a|b$ the fact that a divides b

- F is surjective $\Leftrightarrow$ $\gcd(s, \lambda_{-r}, \dots, \lambda_r) = 1$
- F is injective $\Leftrightarrow$ $\forall p \in \mathcal{P}\ \exists!\ \lambda_i$ t.c. $p \nmid \lambda_i$
- F is transitive $\Leftrightarrow$ $\gcd(s, \lambda_{-r}, \dots, \lambda_{-1}, \lambda_1, \dots, \lambda_r) = 1$
- F is sensitive $\Leftrightarrow$ $\exists p \in \mathcal{P} : p \nmid \gcd(\lambda_{-r}, \dots, \lambda_{-1}, \lambda_1, \dots, \lambda_r)$
- F is expansive $\Leftrightarrow$ $\gcd(\lambda_{-r}, \dots, \lambda_{-1}) = \gcd(\lambda_1, \dots, \lambda_r) = 1$
- F is surjective $\Leftrightarrow$ F has DPO
- F is equicontinuous $\Leftrightarrow$ F is almost ep $\Leftrightarrow$ F is not sensitive
- F is chaotic $\Leftrightarrow$ F is transitive

COROLLARY

All the above mentioned properties are decidable for LINEAR CA