Graph entropy

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Oultline

Fixed points of FDSs

Network coding

Entropy of random variables

Graph entropy

Shannon entropy of graphs

Outline

Finite Dynamical Systems

A Finite Dynamical System (FDS) is a function $f:[q]^n \to [q]^n$, where $[q] = \{0, 1, \ldots, q-1\}$.

We denote $f = (f_1, \ldots, f_n)$ where $f_v : [q]^n \to [q]$.

We refer to $x = (x_1, \ldots, x_n) \in [q]^n$ as a state where $x_v \in [q]$.

A fixed point of f is a state x s.t.

$$f(x) = x.$$

The guessing number of f:

$$g(f) = \log_q |\mathrm{Fix}(f)|.$$

Interaction graph

The function f can be represented by its interaction graph $\mathrm{IG}(f)$, with n vertices and where (u,v) is an arc iff f_v depends on x_u.

For any digraph D, the guessing number of D:

$$g(D,q) := \max\{g(f) : f \in F(D,q)\},$$
$$F(D,q) = \{f : [q]^n \to [q]^n : \mathrm{IG}(f) \subseteq D\},$$
$$g(f) = \log_q |\mathrm{Fix}(f)|.$$

The guessing number was introduced by Riis (Riis 06, Riis 07) in his study of Network Coding.

The feedback bound

Let $\tau(D)$ denote the size of a minimum feedback vertex set (complement of a maximum induced acyclic subgraph).

Theorem. For all D and q,

$$g(D,q) \leq \tau(D).$$

Proof. Let I a MFVS and $J = V \setminus I$ in topological order. Let $x, y \in \mathrm{Fix}(f)$ with $x_I = y_I$. Then

$$\begin{aligned} x_{j_1} &= y_{j_1} = f_{j_1}(x_I) \\ x_{j_2} &= y_{j_2} = f_{j_2}(x_I, x_{j_1}), \\ &\vdots \\ x_{j_k} &= y_{j_k} = f_{j_k}(x_I, x_{J-j_k}), \end{aligned}$$

and hence $x = y$.

Outline

The butterfly network

The same message (bit) is sent on all outgoing links

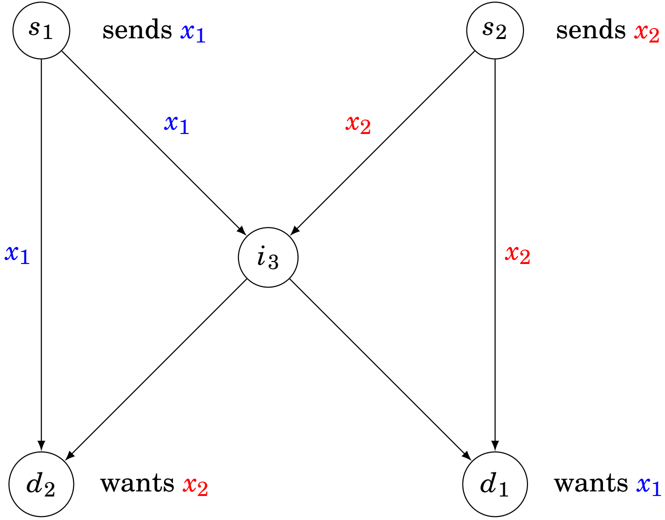


Figure : Butterfly network

The butterfly network

The same message (bit) is sent on all outgoing links

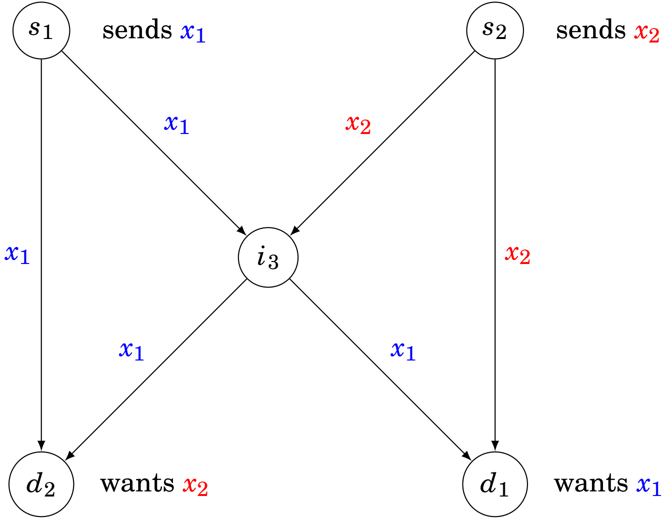


Figure : Butterfly network: Routing

The butterfly network

The same message (bit) is sent on all outgoing links

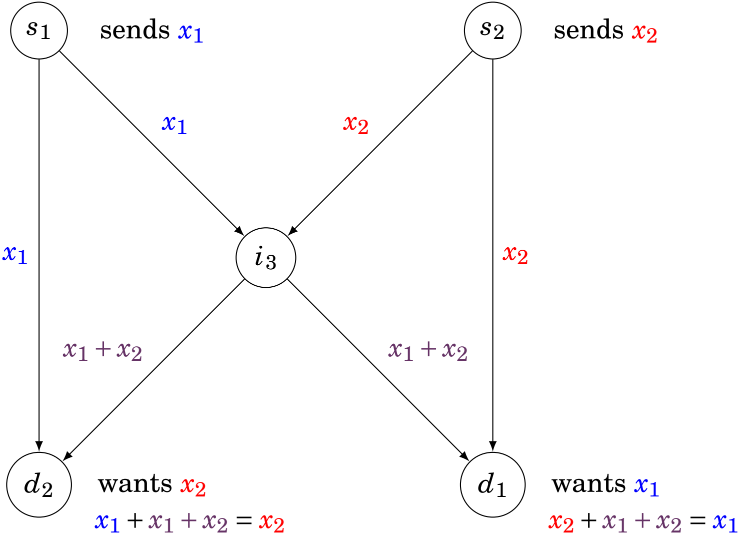


Figure : Butterfly network: Network coding

Network coding and guessing number

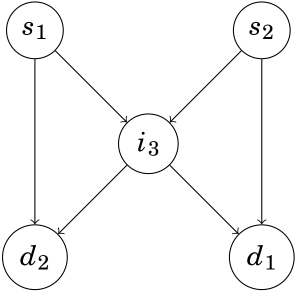
Network coding solvability: can all messages be transmitted at the same time?

Riis converts this problem in terms of guessing number.

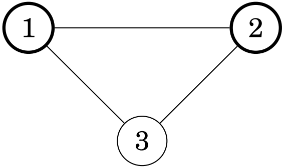
The question becomes: for a digraph D associated to the original Network coding instance,

$$g(D, q) = \tau(D)?$$

Butterfly network and guessing number



(a) Butterfly network



(b) Guessing number of K_3

In general, $g(K_n, q) = n - 1$ and hence

$$g(D, q) \geq n - \pi(D),$$

the clique cover number of D.

Outline

Entropy of random variables

Let X be a discrete random variable with support $\mathscr{X}$ and distribution $p_X(x)$ for all $x \in \mathscr{X}$.

The q-ary entropy of X:

$$H_q(X) = -\sum_{x \in \mathscr{X}} p_X(x) \log_q p_X(x),$$

The entropy is the information (uncertainty) content of X. We have

$$0 \leq H_q(X) \leq \log_q |\mathscr{X}|,$$

where $H(X) = 0$ iff X is deterministic and $H(X) = \log_q |\mathscr{X}|$ iff X is uniform.

Definitions

The joint entropy of a pair of variables (X, Y):

$$H(X,Y) := -\sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}} p_{X,Y}(x,y)\log p_{X,Y}(x,y).$$

We have

$$H(X,Y) = H(Y,X) \geq \max\{H(X), H(Y)\}.$$

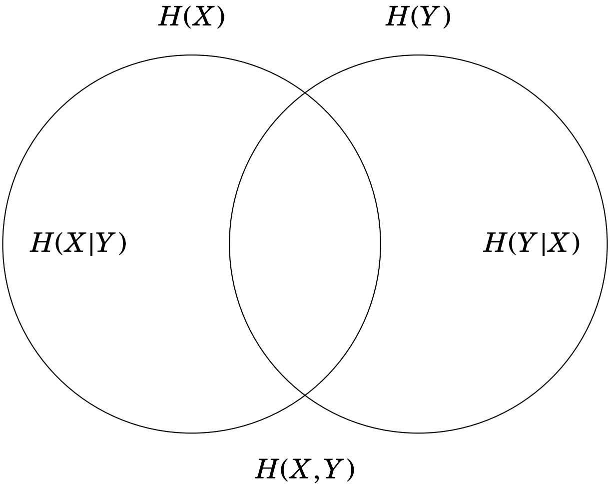
The conditional entropy of Y given X:

$$H(Y|X) := \sum_{x\in\mathcal{X}} p_X(x) H(Y|X=x).$$

We have

$$H(X,Y) = H(X) + H(Y|X) \leq H(X) + H(Y).$$

Venn diagram of entropy



Sub-additivity of entropy

Let $X_1, \ldots, X_n$ a collection of n variables. Build 2^n variables, one for each $S = \{s_1, \ldots, s_k\} \subseteq V$:

$$X_S = (X_{s_1}, \ldots, X_{s_k}).$$

($X_\emptyset$ has entropy 0.)

Denote $H(S) := H(X_S)$ for all $S \subseteq V$.

Theorem. The entropy is sub-additive, i.e. for all $S, T \subseteq V$

$$H(S \cup T) \leq H(S) + H(T).$$

Proof. $H(S \cup T) = H(X_S, X_T) \leq H(X_S) + H(X_T) = H(S) + H(T).$

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Definition

The idea is to study a uniformly chosen fixed point of f:

$$X^f \sim p_{X^f}(x) = \frac{1}{|\mathrm{Fix}(f)|} \quad \forall x \in \mathrm{Fix}(f).$$

Then $H(X^f) = g(f)$.

Let $x \in \mathrm{Fix}(f)$. If we know $x_{N_{\mathrm{in}}(v)}$, then we know $x_v = f_v(x_{N_{\mathrm{in}}(v)})$.

Hence $H(X^f_v | X^f_{N_{\mathrm{in}}(v)}) = 0$ and

$$H(X^f_{N_{\mathrm{in}}(v)\cup v}) = H(X^f_{N_{\mathrm{in}}(v)}) + H(X^f_v | X^f_{N_{\mathrm{in}}(v)}) = H(X^f_{N_{\mathrm{in}}(v)}).$$

The q-ary entropy of D is

$$H(D,q) := \sup H_q(V),$$

sup taken over all families of variables $\{X_v : v \in V\}$, each over $[q]$, s.t.

$$H(N_{\mathrm{in}}(v) \cup v) = H(N_{\mathrm{in}}(v)) \quad \forall v \in V.$$

Using entropy: First attempt

The naive entropy of D is $\epsilon(D) := \sup h(V)$, supremum taken over all $h : 2^V \to \mathbb{R}$ s.t.

$$
\begin{aligned}
h(v) &\leq 1 && \forall v \in V, \\
h(S) &\leq h(T) && \forall S \subseteq T \subseteq V, \\
h(S \cup T) &\leq h(S) + h(T) && \forall S, T \subseteq V, \\
h(N_{\text{in}}(v) \cup v) &= h(N_{\text{in}}(v)) && \forall v \in V.
\end{aligned}
$$

Theorem (Riis 06, G 14).

$$g(D,q) = H(D,q) \leq \epsilon(D) = \tau(D).$$

Outline

Mutual information

Mutual information between X and Y:

$$I(X;Y) := H(X) + H(Y) - H(X,Y).$$

It is the amount of information that X and Y have in common.

Shannon inequality:

$$I(X;Y) \geq 0.$$

Venn diagram

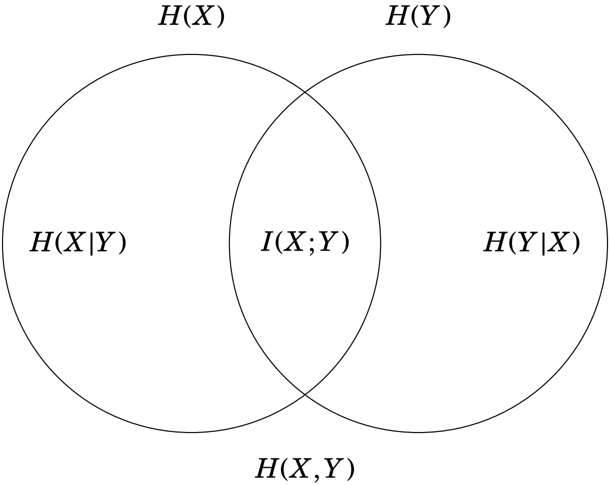

Figure : Venn diagram of entropy

Submodularity of entropy

Theorem. Entropy is submodular, i.e. for all $S, T \subseteq V$,

$$H(S \cup T) + H(S \cap T) \leq H(S) + H(T).$$

Intuition. $X_{S \cap T}$ is common to X_S and X_T, thus

$$\begin{aligned} H(S \cap T) &\leq I(X_S; X_T) \\ &= H(X_S) + H(X_T) - H(X_S, X_T) \\ &= H(S) + H(T) - H(S \cup T). \end{aligned}$$

Shannon entropy of graphs

The Shannon entropy of D is

$$\eta(D) := \sup h(V),$$

sup taken over all functions $h : 2^V \to \mathbb{R}$ s.t.

$$\begin{aligned}
h(v) &\leq 1 && \forall v \in V, \\
h(S) &\leq h(T) && \forall S \subseteq T \subseteq V, \\
h(S \cup T) + h(S \cap T) &\leq h(S) + h(T) && \forall S, T \subseteq V, \\
h(N_{\text{in}}(v) \cup v) &= h(N_{\text{in}}(v)) && \forall v \in V.
\end{aligned}$$

The pentagon

For the pentagon,

$$\tau(C_5) = 3, \quad n - \pi(C_5) = 2, \quad n - \pi^*(C_5) = 2.5.$$

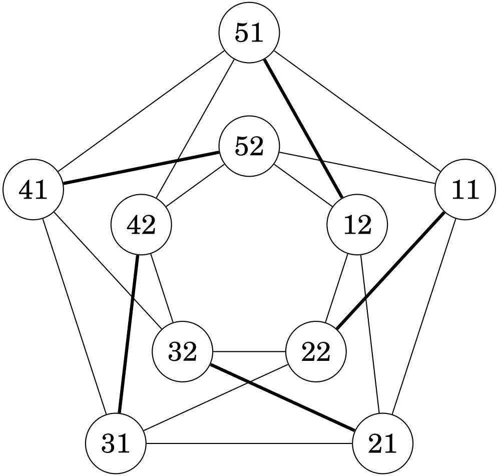

Figure : Optimal fractional clique cover of C_5

Shannon entropy of the pentagon

Let h satisfy the constraints for C_5, then

$$\begin{aligned}
h(V) &= h(2,3,4,5) && \{2,3,4,5\} \text{ is a feedback vertex set,}\\
h(V) &= h(1,3,4) && \{1,3,4\} \text{ is a feedback vertex set,}\\
&\le h(1) + h(3,4) && \text{sub-additivity,}\\
2h(V) &\le h(1) + h(3,4) + h(2,3,4,5) && \text{summation,}\\
&\le h(1) + h(2,3,4) + h(3,4,5) && \text{submodularity,}\\
&= h(1) + h(2,4) + h(3,5) && \{2,4\} = N_{\text{in}}(3) \text{ and } \{3,5\} = N_{\text{in}}(4),\\
&\le 5 && h(v) \le 1 \text{ for all } v \in V.
\end{aligned}$$

Thus $h(V) \le 2.5$.

Further results based on graph entropy

Two extensions in (Christofides and Markström 11).

For odd cycles C_{2k+1} $(k \geq 2)$,

$$\sup_{q\geq 2} g(C_{2k+1}, q) = \eta(C_{2k+1}) = k + \frac{1}{2} < \tau(C_{2k+1}) = k+1.$$

For their complements,

$$\sup_{q\geq 2} g(\overline{C_{2k+1}}, q) = \eta(\overline{C_{2k+1}}) = 2k - 1 - \frac{1}{k} < \tau(\overline{C_{2k+1}}) = 2k-1.$$

And that's it!

Other results on the guessing number

Using non-Shannon inequalities (Baber, Christofides, Dang, Riis, Vaughan 14)

Guessing graph (G + Riis 11): relation to coding theory, extension to signed interaction graphs in (G + Richard + Riis 15)

Approach based on closure solvability: links with matroid theory and secret sharing (G 13, G 14)

System reduction and linear network coding solvability (G + Richard + Fanchon, 15+)