Introduction to Finite Dynamical Systems

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Outline

Outline

Finite Dynamical Systems

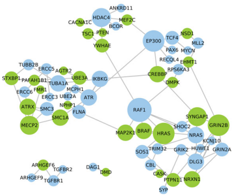
Number of fixed points

Number of periodic points

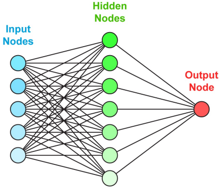
Finite Dynamical Systems

One model for all

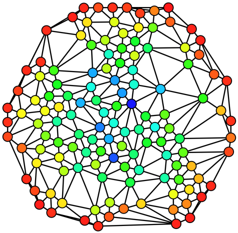
- Network of entities
- Each has a finite dynamic state



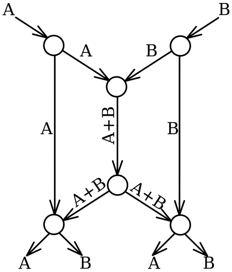
Gene network



Neural network

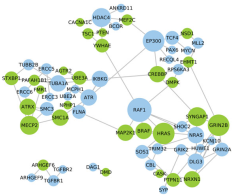


Social network



Network coding

Finite Dynamical Systems



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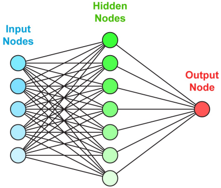
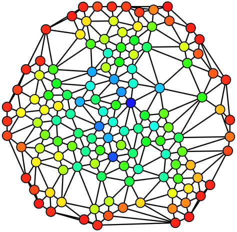


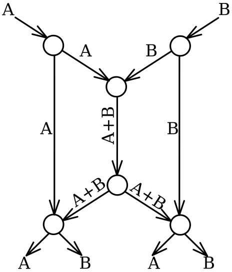
Neural network

A lot to understand

- Work still limited
- No systematic view

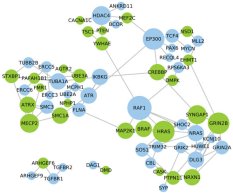


Social network

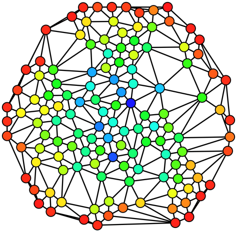


Network coding

Finite Dynamical Systems



Gene network



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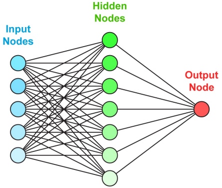
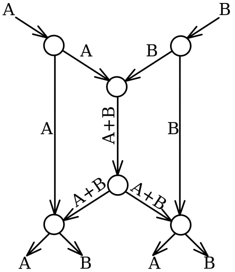
Plan: unified theory of FDSs

- Fundamental results
- Many applications



Neural network



Network coding

The model

The main idea behind Finite Dynamical Systems (FDS) is to have a simple model which can be used in many different situations.

Objective: focus on fundamental phenomena and obtain solid and versatile results.

Formally, let $q \geq 2$ be an integer, $[q] = \{0, 1, \ldots, q-1\}$ be the **alphabet** and n be a positive integer, the number of **entities**. We let $V = \{1, \ldots, n\}$ and we denote $x = (x_1, \ldots, x_n) \in [q]^n$.

Then an FDS is any function $f = (f_1, \ldots, f_n) : [q]^n \to [q]^n$.

Main problem: given some knowledge of f, what can we say about its dynamics?

Parameters of an FDS

1. The **interaction graph** $\mathrm{IG}(f)$, which indicates the underlying network of interactions. Formally, vertices: V and arc (u,v) if and only if $f_v(x)$ essentially depends on x_v:

$$\exists a, b \in [q]^n : a_u \neq b_u,\, a_i = b_i \forall i \neq u,\, f_v(a) \neq f_v(b).$$

   We will focus on FDSs with a prescribed interaction graph:

$$\mathrm{F}[D,q] = \{f : [q]^n \to [q]^n,\, \mathrm{IG}(f) = D\},$$
$$\mathrm{F}(D,q) = \{f : [q]^n \to [q]^n,\, \mathrm{IG}(f) \subseteq D\}.$$

2. The nature of the local functions f_v. We can restrict ourselves to f_v being linear, monotone, or threshold, etc.
3. The alphabet size q. The Boolean case ($q = 2$) is the most commonly used, but it may be degenerate in some cases.
4. The update schedule. Today: parallel schedule only, where entities update their state all at the same time (and x becomes $f(x)$).

Outline

Unique fixed point for acyclic interaction graph

A **fixed point** of f is $x \in [q]^n$ such that $f(x) = x$.

Theorem. If $\mathrm{IG}(f)$ is acyclic, then f has a unique fixed point.

Proof. Sort the vertices in topological order: $(i, j) \in D = \mathrm{IG}(f)$ only if $i < j$. Then y is the unique fixed point of f, where

$$\begin{aligned}
y_1 &= f_1 \\
y_2 &= f_2(y_1) \\
&\vdots \\
y_n &= f_n(y_1, \ldots, y_{n-1}).
\end{aligned}$$

Therefore, in order to have multiple fixed points, we need feedback (cycles in IG).

The feedback bound

A **feedback vertex set** is a set I of vertices such that $D - I$ is acyclic. Let τ denote the minimum size of a FVS of D.

It is useful to use the logarithm: let

$$\mathrm{fix}(f) = \log_q |\mathrm{Fix}(f)|,$$
$$\mathrm{fix}(D,q) = \max\{\mathrm{fix}(f) : f \in \mathrm{F}(D,q)\}.$$

Theorem (The feedback bound). For any D and any q,

$$\mathrm{fix}[D,q] \leq \mathrm{fix}(D,q) \leq \tau(D).$$

Proof. Let I be an FVS of D. Once we fix the value in I, we obtain an acyclic digraph, with one fixed point. Hence $|\mathrm{Fix}(f)| \leq q^{|I|}$.

Lower bound by packing cycles

Let $\vec{C}_n$ denote the (directed) cycle on n vertices.

Proposition. $\mathrm{fix}(\vec{C}_n, q) = 1$.

Let $\nu(D)$ denote the maximum number of disjoint cycles in D.

Corollary. For any D and q,

$$\mathrm{fix}(D, q) \geq \nu(D).$$

Note that we sometimes have $\mathrm{fix}[D, q] < \nu(D)$.

Quick review of known results

Determining $\mathrm{fix}[D,q]$ or $\mathrm{fix}(D,q)$ is far from easy...

Some results about reaching the feedback bound:

1. (Folklore) $\mathrm{fix}[D,q] = 1$ if $\tau(D) = 1$.
2. (Shenvi and Dey 10) $\mathrm{fix}(D,q) = 2$ if $\tau(D) = 2$.
3. (Riis 06-07) The feedback bound is reached for perfect graphs.
4. (Riis 06; Christofides and Markström 11) The feedback bound is not reached for odd undirected cycles. E.g. $\mathrm{fix}(C_5,q) \leq 2.5$.
5. (Aracena, Richard and Salinas 15) The feedback bound is reached by monotone functions if and only if $\nu(D) = \tau(D)$.
6. (G, Richard and Fanchon 15) The feedback bound is reached by linear functions on triangle-free undirected graphs if and only if $\nu(D) = \tau(D)$.

Outline

Images and periodic points

A **periodic point** of f is x such that $f^k(x) = x$ for some $k \geq 1$.

The **rank** of f is simply the number of its images; the **periodic rank** of f is the number of its periodic points. Again, it will be useful to take the logarithm:

$$\mathrm{ima}(f) = \log_q |\mathrm{Ima}(f)|, \quad \mathrm{per}(f) = \log_q |\mathrm{Per}(f)|.$$

Note that

$$\mathrm{per}(f) = \lim_{k \to \infty} \mathrm{ima}(f^k) = \mathrm{ima}(f^{q^n}).$$

Clearly,

$$\mathrm{Fix}(f) \subseteq \mathrm{Per}(f) \subseteq \mathrm{Ima}(f), \quad \mathrm{fix}(f) \leq \mathrm{per}(f) \leq \mathrm{ima}(f).$$

Theorem. If $\mathrm{IG}(f)$ is acyclic, then $\mathrm{per}(f) = 0$.

Maximum number of periodic points I

So, once again, it is the presence of cycles that yields multiple periodic points. But not in the same way...

Let $W = (w_0, w_1, \ldots, w_k)$ and $W' = (w'_0, w'_1, \ldots, w'_k)$ be two k-walks on D (i.e. (w_i, w_{i+1}) and (w'_i, w'_{i+1}) are arcs in D).

We say that W and W' are **independent** if $w_i \neq w'_i$ for all $0 \leq i \leq k$. Let $\alpha_k(D)$ be the largest number of pairwise independent k-walks as .

Theorem. (Poljak 89) $\alpha_k(D)$ is the maximum number of independent paths obtained from disjoint cycles and paths:

$$\alpha_k(D) = \max \left\{ \sum_i |C_i| + \sum_j (|P_j| - k) \right\}.$$

In particular, if $k \geq n$, then

$$\alpha_k(D) = \alpha_n(D) = \max \sum_i |C_i|.$$

Maximum number of periodic points II

Theorem. (G 16+) For any k and any $q \geq 2$,

$$\max\{\mathrm{ima}(f^k) : f \in \mathrm{F}(D,q)\} = \alpha_k(D).$$

Moreover, for any k and any $q \geq 3$,

$$\max\{\mathrm{ima}(f^k) : f \in \mathrm{F}[D,q]\} = \alpha_k(D).$$

Corollary. For any $q \geq 3$,

$$\mathrm{ima}[D,q] = \alpha_1(D), \quad \mathrm{per}[D,q] = \alpha_n(D).$$

In particular, $\mathrm{F}[D,q]$ contains a permutation of $[q]^n$ if and only if the vertices of D can be covered by disjoint cycles.

Idea of proof

Lower bound: Use the disjoint cycles and paths.

1. For $\mathrm{F}(D,q)$, this works directly.
2. For $\mathrm{F}[D,q]$, use a technical trick to remove the influence of the other arcs.

Upper bound: Convert the problem to use a result in (Riis and Gadouleau 11).

1. Construct a blow-up graph D_k. Vertices are $k+1$ copies of V: $V_0, V_1, \ldots, V_k$. Edges of the form (u_i, v_{i+1}) where $(u,v) \in D$.
2. Disjoint paths in D_k correspond to independent k-paths in D.
3. Max-flow min-cut: there exists S of size $\alpha_k(D)$ which disconnects V_0 and V_k.
4. Then f^k “depends” on the variables of S, hence $\mathrm{ima}(f^k) \leq |S|$.

Comments and extensions

For $q = 2$, there are graphs D whose vertices can be covered by disjoint cycles, yet $\mathrm{F}[D,q]$ contains no permutations.
Can we characterise these "troublesome" graphs?
Can we obtain meaningful lower bounds on the asymptotic rank?

The main theorem is the analogue of Poljak's theorem on the maximum rank of a real matrix with a given support (Poljak 89).

The average rank also tends to α_1:

$$\lim_{q\to\infty} \frac{1}{|\mathrm{F}[D,q]|} \sum_{f\in\mathrm{F}[D,q]} \mathrm{ima}(f) = \alpha_1(D).$$

There is no counterpart for the average periodic rank.
But can we still obtain meaningful results?