Fixing monotone Boolean networks asynchronously

Maximilien Gadouleau
Joint work with Julio Aracena, Adrien Richard and Lilian Salinas

Durham University

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Boolean networks

An n-component Boolean network (n-network for short) is a mapping

$$f : \{0,1\}^n \to \{0,1\}^n.$$

We view $x = (x_1, \ldots, x_n) \in \{0,1\}^n$.

Similarly, we view $f = (f_1, \ldots, f_n)$, where $f_i : \{0,1\}^n \to \{0,1\}$.

Example of a 3-network:

x	$f(x)$
000	000
001	000
010	001
011	001
100	010
101	000
110	010
111	100

$$f_1(x) = x_1 \wedge x_2 \wedge x_3$$
$$f_2(x) = x_1 \wedge \neg x_3$$
$$f_3(x) = x_2 \wedge \neg x_1.$$

Asynchronous dynamics

We are interested in the asynchronous dynamics of a network f, i.e. each event is of the form

$$x = (x_1, \dots, x_i, \dots, x_n) \quad \mapsto \quad f^i(x) := (x_1, \dots, f_i(x), \dots, x_n).$$

So we consider any word $w = w_1 \dots w_k$ over the alphabet $\{1, \dots, n\}$ and the corresponding

$$f^w = f^{w_k} \circ f^{w_{k-1}} \circ \dots \circ f^{w_1}.$$

Asynchronous graph

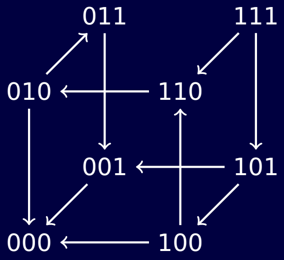
The asynchronous dynamics are neatly represented via the asynchronous graph $\Gamma(f)$ of f.

The vertex set of $\Gamma(f)$ is $\{0,1\}^n$, and xy is an arc iff

$$x = y + e^i \qquad \text{and} \qquad f_i(x) = y_i \qquad \text{for some } i.$$

Example.

x	$f(x)$
000	000
001	000
010	001
011	001
100	010
101	000
110	010
111	100



Fixable networks

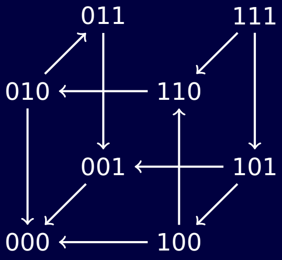
A fixed point of f is y such that $f(y) = y$.

We say that x can reach y if there is a path in $\Gamma(f)$ from x to y.
This is equivalent to: $f^w(x) = y$ for some word w.

We say f is fixable if the following equivalent conditions hold:

1. any state can reach a fixed point;
2. some word w fixes f, i.e. $f^w(x)$ is a fixed point for all x.

In our example, $w = 1231$ fixes f.



$$
\begin{array}{lllll}
000 & \xrightarrow{1} 000 & \xrightarrow{2} 000 & \xrightarrow{3} 000 & \xrightarrow{1} 000 \\
001 & \xrightarrow{1} 001 & \xrightarrow{2} 001 & \xrightarrow{3} 000 & \xrightarrow{1} 000 \\
010 & \xrightarrow{1} 010 & \xrightarrow{2} 000 & \xrightarrow{3} 000 & \xrightarrow{1} 000 \\
011 & \xrightarrow{1} 011 & \xrightarrow{2} 001 & \xrightarrow{3} 000 & \xrightarrow{1} 000 \\
100 & \xrightarrow{1} 000 & \xrightarrow{2} 000 & \xrightarrow{3} 000 & \xrightarrow{1} 000 \\
101 & \xrightarrow{1} 001 & \xrightarrow{2} 001 & \xrightarrow{3} 000 & \xrightarrow{1} 000 \\
110 & \xrightarrow{1} 010 & \xrightarrow{2} 000 & \xrightarrow{3} 000 & \xrightarrow{1} 000 \\
111 & \xrightarrow{1} 111 & \xrightarrow{2} 101 & \xrightarrow{3} 100 & \xrightarrow{1} 000
\end{array}
$$

Questions about fixable networks

Question 0. Which networks are fixable?

- Bollobás-Gotsman-Shamir 93. Almost all networks with a fixed point are fixable.
- Special classes of networks are fixable: acyclic interaction graph, symmetric threshold, monotone, balanced, etc.

Question 1. How fast can we fix each fixable network?

- Some networks require words of exponential length to be fixed.
- Which ones can be fixed in polynomial (or cubic, quadratic, linear) time?

Question 2. How fast can we fix entire classes of networks at once?

- We say w fixes a family $\mathcal{F}$ of networks if it fixes every single network in $\mathcal{F}$.
- How much harder is it to fix the whole of $\mathcal{F}$ compared to fixing a particular network in $\mathcal{F}$?

Monotone Boolean networks

We denote $x \leq y$ if $x_i \leq y_i$ for all $1 \leq i \leq n$.

We say f is monotone if $x \leq y$ implies $f(x) \leq f(y)$.

Knaster-Tarski 28-55. The set of fixed points of a monotone network forms a lattice (hence f has a fixed point).

Richard 10. The set of fixed points any x can reach forms a lattice (hence f is fixable).

Richard 10. Any x can reach some fixed point y via a geodesic (i.e. in $d_{\mathrm{H}}(x, y)$ updates).

Our results for monotone networks

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Theorem 1. For any $\epsilon > 0$ and for all n large enough, there exists a monotone n-network which requires words of length at least

$$\left(\frac{1}{e} - \epsilon\right) n^2$$

to be fixed.

Theorem 2. There exists a word of length

$$\frac{1}{3} n^3$$

which fixes all monotone n-networks.

Path networks

A network is a path network if it can be expressed as

$$f_1(x) = 1,$$
$$f_2(x) = x_1,$$
$$\vdots$$
$$f_n(x) = x_{n-1}.$$

Then w fixes f iff it contains $1, 2, \ldots, n$ as a subsequence.

Fixing all path networks

W fixes all the path n-networks iff it contains all permutations of $1, \ldots, n$ as subsequences, i.e. it is n-universal.

Let $h(n)$ be the length of a shortest n-universal sequence.

1. $h(n)$ is not known in general!
2. $h(n) \leq n^2$, since

$$\Omega^n := 1, \ldots, n, 1, \ldots, n, \ldots, 1, \ldots, n$$

   is n-universal.
3. Kleitman-Kwiatkowski 76. $h(n) = (1 - o(1))n^2$.

Recall

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Theorem 1. For any $\epsilon > 0$ and for all n large enough, there exists a monotone n-network which requires words of length at least

$$\left(\frac{1}{e} - \epsilon\right) n^2$$

to be fixed.

Proof of Theorem 1 (1: simulation by monotone)

Let $n = m + c$ with $\{1, ..., n\} = M \cup C$ and consider a family of monotone m-networks: $\mathcal{G} = \{g^1, ..., g^{|\mathcal{G}|}\}$.

If $|\mathcal{G}| \leq \binom{c}{c/2}$, then $\mathcal{G}$ can be simulated by the following monotone n-network as follows.

Let $\phi : \binom{C}{c/2} \to \{1, ..., |\mathcal{G}|\}$ be surjective and

$$f_M(x) = \begin{cases} 1 & \text{if } w_{\mathrm{H}}(x_C) > c/2 \\ g^{\phi(x_C)}(x_M) & \text{if } w_{\mathrm{H}}(x_C) = c/2 \\ 0 & \text{if } w_{\mathrm{H}}(x_C) < c/2 \end{cases}$$

$$f_C(x) = x_C$$

x_C is the "control part", which determines which network in $\mathcal{G}$ is simulated.

Then w fixes f only if it fixes $\mathcal{G}$.

Proof of Theorem 1 (2: simulating paths)

Let $n = m + c$ and $\mathcal{G}$ be the family of all m-path networks.

We have $|\mathcal{G}| = m!$ and we require $m! \leq \binom{c}{c/2}$, thus at best $m = \Theta(n/\log n)$.

Simulating $\mathcal{G}$ then yields f, which requires words of length

$$h(m) = \Theta(m^2) = \Theta(n^2/\log^2 n).$$

Proof of Theorem 1 (3: only a few paths suffice)

Idea. Only a subexponential set of paths are necessary.

Theorem. For any $\delta > 0$ and m sufficiently large, there is a set of at most $m^{m^{\frac{1}{2}+\delta}}$ permutations of M such that any word containing all these permutations as subsequences is of length at least $(\frac{1}{e} - \delta)m^2$.

Using the corresponding family of paths, we can use $n = m + c$ with $c = o(n)$.
This yields f, which requires words of length

$$(\frac{1}{e} - \delta)m^2 = (\frac{1}{e} - \epsilon)n^2.$$

Recall

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Theorem 2. There exists a word of length

$$\frac{1}{3}n^3$$

which fixes all monotone n-networks.

Increasing networks

A network is increasing if $f(x) \geq x$ for all x.

Lemma. The word $\Omega^n = 1, \ldots, n, \ldots, 1, \ldots, n$ fixes all increasing networks.

Proof. Let f be increasing and $g := f^{1,\ldots,n}$.
For any y, if y is not a fixed point of f, then $g(y) > y$.
If $f^{\Omega^n}(x)$ is not a fixed point of f, then

$$x < g(x) < \cdots < f^{\Omega^n}(x) < g\left(f^{\Omega^n}(x)\right)$$

and $w_{\mathrm{H}}\left(g\left(f^{\Omega^n}(x)\right)\right) \geq n+1$.

Theorem. The word ω fixes all increasing networks if and only if it is n-universal.

Proof of Theorem 2

Let ω^k be a k-universal word for any k.

Let W^n be recursively defined as

$$W^1 = 1 \qquad \text{and} \qquad W^n = W^{n-1}, n, \omega^{n-1}.$$

Then W^n fixes all monotone networks.

Idea. By induction hypothesis, W^{n-1} fixes x_1 to x_{n-1}.
Applying the update f^n can lead to three scenarios:

1. if x_n does not change, then we are at a fixed point;
2. if x_n increases from 0 to 1, then f behaves like an increasing network (and x_n is now fixed), hence ω^{n-1} fixes it;
3. if x_n decreases from 1 to 1, then f behaves like an decreasing network (same as before).

Grazie mille !