Packing and Covering Properties of Subspace Codes for Error Control in Random Linear Network Coding

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***Abstract*—Codes in the projective space and codes in the Grassmannian over a finite field—referred to as subspace codes and constant-dimension codes (CDCs), respectively—have been proposed for error control in random linear network coding. For subspace codes and CDCs, a subspace metric was introduced to correct both errors and erasures, and an injection metric was proposed to correct adversarial errors. In this paper, we investigate the packing and covering properties of subspace codes with both metrics. We first determine some fundamental geometric properties of the projective space with both metrics. Using these properties, we then derive bounds on the cardinalities of packing and covering subspace codes, and determine the asymptotic rates of optimal packing and optimal covering subspace codes with both metrics. Our results not only provide guiding principles for the code design for error control in random linear network coding, but also illustrate the difference between the two metrics from a geometric perspective. In particular, our results show that optimal packing CDCs are optimal packing subspace codes up to a scalar for both metrics if and only if their dimension is half of their length (up to rounding). In this case, CDCs suffer from only limited rate loss as opposed to subspace codes with the same minimum distance. We also show that optimal covering CDCs can be used to construct asymptotically optimal covering subspace codes with the injection metric only.**

***Index Terms*—Constant-dimension codes (CDCs), covering, error control codes, injection metric, network coding, packing, random linear network coding, subspace codes, subspace metric.**

I. Introduction

DUE to its vector-space preserving property, random linear network coding [1], [2] can be viewed as transmitting subspaces over an operator channel [3]. As such, error control for random linear network coding can be modeled as a coding problem, where codewords are subspaces and the distance is measured by either the subspace distance [3] or the injection metric [4]. Codes in the projective space, referred to as subspace codes henceforth, and codes in the Grassmannian, referred to as constant-dimension codes (CDCs) henceforth, have been both investigated for error control in random linear network coding. Using CDCs is sometimes advantageous since the fixed dimension of CDCs simplifies the network protocol somewhat [3].

The construction and properties of CDCs thus have attracted a lot of attention. Different constructions of CDCs have been proposed [3], [5]–[7]. Bounds on CDCs based on packing properties are investigated (see, for example, [3], [6], [8], and [9]), and the covering properties of CDCs are investigated in [7]. The construction and properties of subspace codes have received less consideration, and previous works on subspace codes (see, for example, [10]–[12]) have focused on the packing properties. In [10], bounds on the maximum cardinality of a subspace code with the subspace metric, notably the counterpart of the Gilbert bound, are derived. Another bound relating the maximum cardinality of CDCs to that of subspace codes is given in [11]. Bounds and constructions of subspace codes are also investigated in [12]. Despite the previous works, two significant problems remain open. First, despite the aforementioned advantage of CDCs, what is the rate loss of CDCs as opposed to subspace codes of the same minimum distance and hence error correction capability? Since random linear network coding achieves multicast capacity with probability exponentially approaching 1 with the length of the code [1], the asymptotic rates of subspace codes and asymptotic rate loss of CDCs are both significant. The second problem involves the two metrics that have been introduced for subspace codes: What is the difference between the two metrics proposed for subspace codes and CDCs beyond those discussed in [4]? Note that the two questions are somewhat related, since the first question is applicable for both metrics. The answers to these questions are significant to the code design for error control in random linear network coding.

Aiming to answer these two questions, our work in this paper focuses on the packing and covering properties of subspace codes. Packing and covering properties not only are interesting in their own right as fundamental geometric properties, but also are significant for various practical purposes. First, our work is motivated by their significance to design and decoding of subspace codes. Since a code can be viewed as a packing of its ambient space, the significance of packing properties is clear. In contrast, the importance of covering properties is more subtle and deserves more explanation. For example, a class of nearly optimal CDCs, referred to as liftings of rank metric codes, have covering radii no less than their minimum distance and thus are not optimal CDCs [7]. This example shows how a covering property is relevant to the design of subspace codes. The covering radius also characterizes the decoding performance

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of a code, since it is the maximum weight of a decodable error by minimum distance decoding [13] and also has applications to decoding with erasures [14]. Second, covering properties are also important for other reasons. For example, covering properties are important for the security of keystreams against cryptanalytic attacks [15].

Our main contributions of this paper are that for both metrics, we first determine some fundamental geometric properties of the projective space, and then use these properties to derive bounds and to determine the asymptotic rates of subspace codes based on packing and covering. Our results provide some answers to both open problems above. First, our results show that for both metrics optimal packing CDCs are optimal packing subspace codes up to a scalar if and only if their dimension is half of their length (up to rounding), which implies that in this case CDCs suffer from a limited rate loss as opposed to subspace codes with the same minimum distance. Furthermore, when the asymptotic rate of subspace codes is fixed, the relative subspace distance of optimal subspace codes is twice as much as the relative injection distance. Second, our results illustrate the difference between the two metrics from a geometric perspective. Above all, the projective space has different geometric properties under the two metrics. The different geometric properties further result in different asymptotic rates of covering codes with the two metrics. With the injection metric, optimal covering CDCs can be used to construct asymptotically optimal covering subspace codes. However, with the subspace metric, this does not hold.

To the best of our knowledge, our results on the geometric properties of the projective space are novel, and our investigation of covering properties of subspace codes is the first one in the literature. Note that our investigation of covering properties differs from the study in [7]: while how CDCs cover the Grassmannian was investigated in [7], we consider how subspace codes cover the whole projective space in this paper. Our investigation of packing properties leads to tighter bounds than the Gilbert bound in [10], and our relation between the optimal cardinalities of subspace codes and CDCs is also more precise than that in [11]. Our asymptotic rates based on packing properties also appear to be novel.

The rest of the paper is organized as follows. Section II reviews necessary background on subspace codes, CDCs, and related concepts. In Section III, we investigate the packing and covering properties of subspace codes with the subspace metric. In Section IV, we study the packing and covering properties of subspace codes with the injection metric. Finally, Section V summarizes our results and provides future work directions.

II. Preliminaries

We refer to the set of all subspaces of $\mathrm{GF}(q)^n$ with dimension r as the Grassmannian of dimension r and denote it as $E_r(q,n)$; we refer to $E(q,n)=\bigcup_{r=0}^{n} E_r(q,n)$ as the projective space. We have $|E_r(q,n)|=\begin{bmatrix} n \\ r \end{bmatrix}$, where $\begin{bmatrix} n \\ r \end{bmatrix}=\prod_{i=0}^{r-1}\frac{q^n-q^i}{q^r-q^i}$ is the Gaussian binomial [16]. A very instrumental result [17] about the Gaussian binomial is that for all $0\le r\le n$

$$q^{r(n-r)} \le \begin{bmatrix} n \\ r \end{bmatrix} < K_q^{-1} q^{r(n-r)} \tag{1}$$

where $K_q=\prod_{j=1}^{\infty}(1-q^{-j})$ represents the ratio of nonsingular matrices in $\mathrm{GF}(q)^{n\times n}$ as n tends to infinity. By definition, $K_q=\phi(q^{-1})$, where ϕ is the Euler function. Furthermore, by the pentagonal number theorem, $K_q=\sum_{n=-\infty}^{\infty}(-1)^n q^{(n-3n^2)/2}$ [18]. Finally, we also have $K_q^{-1}=\sum_{k=0}^{\infty}p(k)q^{-k}$, where $p(k)$ is the partition number of k [16].

For $U,V\in E(q,n)$, both the *subspace metric* [3, eq. (3)] $d_{\mathrm{S}}(U,V)\stackrel{\mathrm{def}}{=}\dim(U+V)-\dim(U\cap V)$ and *injection metric* [4, Def. 1]

$$\begin{aligned} d_{\mathrm{I}}(U,V) &\stackrel{\mathrm{def}}{=} \frac{1}{2}d_{\mathrm{S}}(U,V)+\frac{1}{2}|\dim(U)-\dim(V)| \\ &= \max\{\dim(U),\dim(V)\}-\dim(U\cap V) \quad (2)\\ &= \dim(U+V)-\min\{\dim(U),\dim(V)\} \\ &\ge |\dim(U)-\dim(V)| \quad (3)\end{aligned}$$

are metrics over $E(q,n)$. For all $U,V\in E(q,n)$

$$\frac{1}{2}d_{\mathrm{S}}(U,V)\le d_{\mathrm{I}}(U,V)\le d_{\mathrm{S}}(U,V) \tag{4}$$

and $d_{\mathrm{I}}(U,V)=\frac{1}{2}d_{\mathrm{S}}(U,V)$ if and only if $\dim(U)=\dim(V)$, and $d_{\mathrm{I}}(U,V)=d_{\mathrm{S}}(U,V)$ if and only if $U\subseteq V$ or $V\subseteq U$. A *subspace code* is a nonempty subset of $E(q,n)$. The minimum subspace (respectively, injection) distance of a subspace code is the minimum subspace (respectively, injection) distance over all pairs of distinct codewords. A subset of $E_r(q,n)$ is called a CDC. A CDC is thus a subspace code whose codewords have the same dimension. Since for CDCs $d_{\mathrm{I}}(U,V)=\frac{1}{2}d_{\mathrm{S}}(U,V)$, we focus on the injection metric when considering CDCs. We denote the maximum cardinality of a CDC in $E_r(q,n)$ with minimum injection distance d as $A_{\mathrm{C}}(q,n,r,d)$. We have $A_{\mathrm{C}}(q,n,r,d)=A_{\mathrm{C}}(q,n,n-r,d)$, $A_{\mathrm{C}}(q,n,r,1)=\begin{bmatrix} n \\ r \end{bmatrix}$ and it is shown [7], [9] for $r\le\left\lfloor\frac{n}{2}\right\rfloor$ and $2\le d\le r$

$$\begin{aligned} q^{(n-r)(r-d+1)}+1 &\le A_{\mathrm{C}}(q,n,r,d) \\ &\le \frac{\begin{bmatrix} n \\ r-d+1 \end{bmatrix}}{\begin{bmatrix} r \\ r-d+1 \end{bmatrix}} < K_q^{-1}q^{(n-r)(r-d+1)}. \quad (5)\end{aligned}$$

The lower bound on $A_{\mathrm{C}}(q,n,r,d)$ in (5) is implicit from the code construction in [7], and the upper bounds on $A_{\mathrm{C}}(q,n,r,d)$ in (5) are from [3]. Thus, CDCs in $E_r(q,n)$ ($r\le\left\lfloor\frac{n}{2}\right\rfloor$) with minimum injection distance d and cardinality $q^{(n-r)(r-d+1)}$ proposed in [3] are optimal up to a scalar; we refer to these CDCs as KK codes henceforth. The covering radius in $E_r(q,n)$ of a CDC $\mathcal{C}$ is defined as $\max_{U\in E_r(q,n)}d_{\mathrm{I}}(U,\mathcal{C})$. We also denote the minimum cardinality of a CDC with covering radius ρ in $E_r(q,n)$ as $K_{\mathrm{C}}(q,n,r,\rho)$ [7]. It was shown in [7] that $K_{\mathrm{C}}(q,n,r,\rho)$ is on the order of $q^{r(n-r)-\rho(n-\rho)}$, and an asymptotically optimal construction of covering CDCs is designed in [7, Prop. 12].

III. Packing and Covering Properties of Subspace Codes With the Subspace Metric

A. Properties of Balls With Subspace Radii

We first investigate the properties of balls with subspace radii in $E(q,n)$, which will be instrumental in our study of packing

and covering properties of subspace codes with the subspace metric. We first derive bounds on $|E(q,n)|$ below. In order to simplify notations, we denote $\theta(q) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} q^{-n^2}$, which is related to the Jacobi theta function $\vartheta_3(z,q) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2niz}$ by $\theta(q) = \frac{1}{2}\left[\vartheta_3(0,q^{-1})+1\right]$ [19]. We remark that $\theta(q) > 1$ for all $q \geq 2$, and that $\theta(q)$ is a decreasing function of q and approaches 1 as q tends to infinity.

Lemma 1: For all n, $q^{\left\lfloor \frac{n}{2} \right\rfloor (n-\left\lfloor \frac{n}{2} \right\rfloor)} \leq |E(q,n)| < 2K_q^{-1}\theta(q) q^{\left\lfloor \frac{n}{2} \right\rfloor (n-\left\lfloor \frac{n}{2} \right\rfloor)}$.

Proof: We have $|E(q,n)| = \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} \geq \begin{bmatrix} n \\ \left\lfloor \frac{n}{2} \right\rfloor \end{bmatrix} \geq q^{\left\lfloor \frac{n}{2} \right\rfloor (n-\left\lfloor \frac{n}{2} \right\rfloor)}$ by (1), which proves the lower bound. Also, $\begin{bmatrix} n \\ n-r \end{bmatrix} = \begin{bmatrix} n \\ r \end{bmatrix}$ and hence $\sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} \leq 2\sum_{r=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \begin{bmatrix} n \\ r \end{bmatrix} < 2K_q^{-1} \sum_{r=0}^{\left\lfloor \frac{n}{2} \right\rfloor} q^{r(n-r)}$ by (1). Therefore

$$\begin{aligned} |E(q,n)| &< 2K_q^{-1} q^{\left\lfloor \frac{n}{2} \right\rfloor (n-\left\lfloor \frac{n}{2} \right\rfloor)} \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor} q^{-i(n-2\left\lfloor \frac{n}{2} \right\rfloor+i)} \\ &< 2K_q^{-1}\theta(q) q^{\left\lfloor \frac{n}{2} \right\rfloor (n-\left\lfloor \frac{n}{2} \right\rfloor)}. \end{aligned}$$

□

We observe that by (1) and Lemma 1, $|E_r(q,n)|$ is the same as $|E(q,n)|$ up to a scalar when $r = \left\lfloor \frac{n}{2} \right\rfloor$ or $r = n - \left\lfloor \frac{n}{2} \right\rfloor$. That is, the volume of $E_{\left\lfloor \frac{n}{2} \right\rfloor}(q,n)$, which is equal to that of $E_{n-\left\lfloor \frac{n}{2} \right\rfloor}(q,n)$ when $\left\lfloor \frac{n}{2} \right\rfloor \neq n - \left\lfloor \frac{n}{2} \right\rfloor$, dominates the volumes of other Grassmannians. This geometric property has significant implication to the packing properties of subspace codes.

We now determine the number of subspaces at a given subspace distance from a fixed subspace. Let us denote the number of subspaces with dimension s at subspace distance d from a subspace with dimension r as $N_{\mathrm{S}}(r,s,d)$.

Lemma 2: $N_{\mathrm{S}}(r,s,d)$ is given by $q^{u(d-u)} \begin{bmatrix} r \\ u \end{bmatrix} \begin{bmatrix} n-r \\ d-u \end{bmatrix}$ when $u = \frac{r+d-s}{2}$ is an integer, and 0 otherwise.

Proof: For $U \in E_r(q,n)$ and $V \in E_s(q,n)$, $d_{\mathrm{S}}(U,V) = d$ if and only if $\dim(U \cap V) = r - u$. Thus, there are $\begin{bmatrix} r \\ u \end{bmatrix}$ choices for $U \cap V$. The subspace V is then completed in $q^{u(d-u)} \begin{bmatrix} n-r \\ d-u \end{bmatrix}$ ways. □

We remark that this result in Lemma 2 is implicitly contained in [10, Th. 5] without an explicit proof. It is formally stated here because it is important to the results in this paper. We also denote the volume of a ball with subspace radius t around a subspace with dimension r as $V_{\mathrm{S}}(r,t) \stackrel{\text{def}}{=} \sum_{d=0}^{t} \sum_{s=0}^{n} N_{\mathrm{S}}(r,s,d)$.

We now derive bounds on the volume of a ball with subspace radius. Since $V_{\mathrm{S}}(r,t) = V_{\mathrm{S}}(n-r,t)$ for all r and t, we only consider $r \leq \left\lfloor \frac{n}{2} \right\rfloor$. Also, we assume $t \leq \left\lfloor \frac{n}{2} \right\rfloor$, for only this case will be needed in this paper.

Proposition 1: For all q, n, $r \leq \left\lfloor \frac{n}{2} \right\rfloor$, and $t \leq \left\lfloor \frac{n}{2} \right\rfloor$, $q^{-\frac{3}{4}} q^{g(r,t)} \leq V_{\mathrm{S}}(r,t) < 2\theta(q^3) K_q^{-2} (1+q^{-\frac{4}{3}}) \theta(q^{\frac{3}{4}}) q^{g(r,t)}$, where

$$g(r,t) = \begin{cases} t(n-r-t), & \text{for } t \leq \frac{n-2r}{3} \\ \frac{1}{12}(n-2r)^2 + \frac{1}{4}t(2n-t), & \text{for } \frac{n-2r}{3} < t \leq \frac{n+4r}{3} \\ (t-r)(n-t+r), & \text{for } \frac{n+4r}{3} < t \leq \frac{n}{2}. \end{cases}$$

The proof of Proposition 1 is given in part A of the Appendix. We remark that the lower and upper bounds on $V_{\mathrm{S}}(r,t)$ in Proposition 1 are tight up to a scalar, and that $g(r,t)$ depends on both r and t. We also observe that $g(r,t)$ decreases with r for $r \leq \left\lfloor \frac{n}{2} \right\rfloor$. That is, the volume of a ball around a subspace of dimension r ($r \leq \left\lfloor \frac{n}{2} \right\rfloor$) decreases with r. This observation is significant to the covering properties of subspace codes with the subspace metric. Fig. 1, where we show $\log_2 [V_{\mathrm{S}}(q,n,r,t)]$ for $q = 2$, $n = 10$, $0 \leq r \leq 5$, and $0 \leq t \leq 5$, illustrates this observation.

B. Packing Properties of Subspace Codes With the Subspace Metric

We are interested in packing subspace codes used with the subspace metric. The maximum cardinality of a code in $E(q,n)$ with minimum subspace distance d is denoted as $A_{\mathrm{S}}(q,n,d)$. Since $A_{\mathrm{S}}(q,n,1) = |E(q,n)|$, we assume $d \geq 2$ henceforth.

We can relate $A_{\mathrm{S}}(q,n,d)$ to $A_{\mathrm{C}}(q,n,r,d)$. First, we remark that $\max_{0 \leq r \leq n} A_{\mathrm{C}}(q,n,r,d) = A_{\mathrm{C}}(q,n,\left\lfloor \frac{n}{2} \right\rfloor, d)$ for all q, n, and $d \leq \left\lfloor \frac{n}{2} \right\rfloor$. The claim is obvious for $d = 1$, and easily shown for $d > 1$ by using (1). We also remark that $A_{\mathrm{C}}(q,n,\left\lceil \frac{n}{2} \right\rceil, d) = A_{\mathrm{C}}(q,n,\left\lfloor \frac{n}{2} \right\rfloor, d)$. For all $J \subseteq \{0,1,\ldots,n\}$, we denote the maximum cardinality of a code with minimum subspace distance d and codewords having dimensions in J as $A_{\mathrm{S}}(q,n,d,J)$. For $2 \leq d \leq 2\left\lfloor \frac{n}{2} \right\rfloor$, $R_d \stackrel{\text{def}}{=} \left\{\left\lceil \frac{d}{2} \right\rceil, \left\lceil \frac{d}{2} \right\rceil + 1, \ldots, n - \left\lceil \frac{d}{2} \right\rceil\right\}$. Proposition 2 compares $A_{\mathrm{S}}(q,n,d)$ to $A_{\mathrm{C}}(q,n,\left\lfloor \frac{n}{2} \right\rfloor, d)$ and shows that $A_{\mathrm{S}}(q,n,d,R_d)$ is a good approximate of $A_{\mathrm{S}}(q,n,d)$.

Proposition 2: For $n = d = 2\left\lfloor \frac{n}{2} \right\rfloor + 1$, $A_{\mathrm{S}}(q, 2\left\lfloor \frac{n}{2} \right\rfloor + 1, 2\left\lfloor \frac{n}{2} \right\rfloor + 1) = 2$ and for $2 \leq d \leq 2\left\lfloor \frac{n}{2} \right\rfloor$, $A_{\mathrm{S}}(q,n,d) \leq A_{\mathrm{S}}(q,n,d,R_d) + 2$. Also, we have

$$\begin{aligned} A_{\mathrm{C}}\left(q,n,\left\lfloor \frac{n}{2} \right\rfloor, \left\lceil \frac{d}{2} \right\rceil\right) &\leq A_{\mathrm{S}}(q,n,d) \\ &\leq 2 + \sum_{r \in R_d} A_{\mathrm{C}}\left(q,n,r,\left\lceil \frac{d}{2} \right\rceil\right). \end{aligned}$$

Proof: Let $\mathcal{C}$ be a code in $E(q,n)$ with minimum subspace distance d. For $C, D \in \mathcal{C}$, we have $\dim(C) + \dim(D) \geq d_{\mathrm{S}}(C,D) \geq d$; therefore, there is at most one codeword with dimension less than $\frac{d}{2}$. Similarly, $\dim(C) + \dim(D) \leq 2n - d_{\mathrm{S}}(C,D) \leq 2n - d$, therefore there is at most one codeword with dimension greater than $\frac{2n-d}{2}$. Thus, $A_{\mathrm{S}}(q,n,d) \leq A_{\mathrm{S}}(q,n,d,R_d) + 2$ for $d \leq 2\left\lceil \frac{n}{2} \right\rceil$ and $A_{\mathrm{S}}(q, 2\left\lfloor \frac{n}{2} \right\rfloor + 1, 2\left\lfloor \frac{n}{2} \right\rfloor + 1) \leq 2$. Since the code $\{\{\mathbf{0}\}, \mathrm{GF}(q)^{2\left\lfloor \frac{n}{2} \right\rfloor + 1}\}$ has minimum subspace distance $2\left\lfloor \frac{n}{2} \right\rfloor + 1$, we obtain $A_{\mathrm{S}}(q, 2\left\lfloor \frac{n}{2} \right\rfloor + 1, 2\left\lfloor \frac{n}{2} \right\rfloor + 1) = 2$.

A CDC in $E_r(q,n)$ with minimum injection distance $\left\lceil \frac{d}{2} \right\rceil$ has minimum subspace distance $\geq d$, and hence $A_{\mathrm{C}}(q,n,r,\left\lceil \frac{d}{2} \right\rceil) \leq A_{\mathrm{S}}(q,n,d)$ for all r. Also, the codewords with dimension r in a code with minimum subspace distance d form a CDC in $E_r(q,n)$ with minimum injection distance at least $\left\lceil \frac{d}{2} \right\rceil$, and hence $A_{\mathrm{S}}(q,n,d) \leq A_{\mathrm{S}}(q,n,d,R_d) + 2 \leq 2 + \sum_{r \in R_d} A_{\mathrm{C}}(q,n,r,\left\lceil \frac{d}{2} \right\rceil)$. □

We compare our lower bound on $A_{\mathrm{S}}(q,n,d)$ in Proposition 2 to the Gilbert bound in [10, Th. 5]. The latter shows that $A_{\mathrm{S}}(q,n,d) \geq \frac{|E(q,n)|}{\mathrm{avg}\{V_{\mathrm{S}}(r,d-1)\}}$, where the average volume $\mathrm{avg}\{V_{\mathrm{S}}(r,d-1)\}$ is taken over all subspaces in $E(q,n)$. Using the bounds on $V_{\mathrm{S}}(r,d-1)$ in Proposition 1, it can be shown that this lower bound

is at most $2K_q^{-1}\theta(q)q^{\frac{3}{4}}q^{\lfloor\frac{n}{2}\rfloor(n-\lfloor\frac{n}{2}\rfloor)-\frac{1}{4}(d-1)(2n-d+1)}$. On the other hand, Proposition 2 and (5) yield $A_{\rm S}(q,n,d) \geq q^{\lfloor\frac{n}{2}\rfloor(n-\lfloor\frac{n}{2}\rfloor)-\frac{1}{4}(d-1)(n+1)}$. The ratio between our lower bound and the Gilbert bound is hence at least $\frac{1}{2}K_q\theta(q)^{-1}q^{\frac{1}{4}(d-1)(n-d)} \geq 1$ for all n and d. Therefore, our lower bound in Proposition 2 is tighter than the Gilbert bound in [10, Th. 5].

The lower bound in Proposition 2 is further tightened below by considering the union of CDCs in different Grassmannians.

Proposition 3: For all q, n, and $2 \leq d \leq n$, we have $A_{\rm S}(q,n,d) \geq \sum_{i=-z}^{z} A_{\rm C}(q,n,\lfloor\frac{n}{2}\rfloor - id, \lceil\frac{d}{2}\rceil)$, where $z = \left\lfloor \frac{\lfloor\frac{n}{2}\rfloor}{d} \right\rfloor$.

Proof: For $i = -z, -z+1, \ldots, z$, let $\mathcal{C}_i$ be a CDC in $E_{\lfloor\frac{n}{2}\rfloor-id}(q,n)$ with minimum subspace distance $2\lceil\frac{d}{2}\rceil$ and cardinality $A_{\rm C}(q,n,\lfloor\frac{n}{2}\rfloor-id,\lceil\frac{d}{2}\rceil)$ and let $\mathcal{C} = \bigcup_{i=-z}^{z}\mathcal{C}_i$. We have $|\mathcal{C}| = \sum_{i=-z}^{z}|\mathcal{C}_i|$, and we now prove that $\mathcal{C}$ has minimum subspace distance at least d by considering two distinct codewords $C_i^j \in \mathcal{C}_i$ and $C_a^b \in \mathcal{C}_a$. First, if $i \neq a$, then $d_{\rm S}(C_i^j, C_a^b) \geq |i-a|d \geq d$; second, if $i = a$ and $j \neq b$, then $d_{\rm S}(C_a^j, C_a^b) \geq 2\lceil\frac{d}{2}\rceil$ by the minimum distance of $\mathcal{C}_a$. □

In order to characterize the rate loss by using CDCs instead of subspace codes, we now compare the cardinalities of optimal subspace codes and optimal CDCs with the same minimum subspace distance d. Note that the bounds on the cardinalities of optimal CDCs in (5) assume the injection metric for CDC. When d is even, a CDC with a minimum subspace distance d has a minimum injection distance $\frac{d}{2}$. When d is odd, a CDC with a minimum subspace distance $d+1$ has a minimum injection distance $\frac{d+1}{2} = \lceil\frac{d}{2}\rceil$. Thus, a CDC has a minimum subspace distance at least d if and only if it has minimum injection distance at least $\lceil\frac{d}{2}\rceil$. Hence, we compare $A_{\rm S}(q,n,d)$ and $A_{\rm C}(q,n,r,\lceil\frac{d}{2}\rceil)$ in Proposition 4.

Proposition 4 (Comparison Between Optimal Subspace Codes and CDCs in the Subspace Metric): For $2 \leq d \leq 2\lfloor\frac{n}{2}\rfloor$ and $\lceil\frac{d}{2}\rceil \leq r \leq \lfloor\frac{n}{2}\rfloor$

$$\begin{aligned} &K_q q^{(\lfloor\frac{n}{2}\rfloor-r)(\lfloor\frac{n}{2}\rfloor-r+\lceil\frac{d}{2}\rceil-1)} A_{\rm C}\left(q,n,r,\left\lceil\frac{d}{2}\right\rceil\right) \\ &< A_{\rm S}(q,n,d) \\ &< 2K_q^{-1}\theta(q)q^{(\lfloor\frac{n}{2}\rfloor-r)(\lfloor\frac{n}{2}\rfloor-r+\lceil\frac{d}{2}\rceil-1)} A_{\rm C}\left(q,n,r,\left\lceil\frac{d}{2}\right\rceil\right). \end{aligned} \tag{6}$$

Proof: By (1), Proposition 2, and (5), we have

$$\begin{aligned} A_{\rm S}(q,n,d) &\geq A_{\rm C}\left(q,n,\left\lfloor\frac{n}{2}\right\rfloor,\left\lceil\frac{d}{2}\right\rceil\right) \\ &\geq q^{(n-\lfloor\frac{n}{2}\rfloor)(\lfloor\frac{n}{2}\rfloor-\lceil\frac{d}{2}\rceil+1)} \\ &> K_q q^{(\lfloor\frac{n}{2}\rfloor-r)(\lfloor\frac{n}{2}\rfloor-r+\lceil\frac{d}{2}\rceil-1)} A_{\rm C}\left(q,n,r,\left\lceil\frac{d}{2}\right\rceil\right). \end{aligned}$$

Also, Proposition 2 and (1) also lead to

$$\begin{aligned} A_{\rm S}(q,n,d) &< 2 + 2K_q^{-1}\sum_{r=\lceil\frac{d}{2}\rceil}^{\lfloor\frac{n}{2}\rfloor} q^{(n-r)(r-\lceil\frac{d}{2}\rceil+1)} \\ &= 2 + 2K_q^{-1}q^{(n-\lfloor\frac{n}{2}\rfloor)(\lfloor\frac{n}{2}\rfloor-\lceil\frac{d}{2}\rceil+1)} \\ &\quad \sum_{i=0}^{\lfloor\frac{n}{2}\rfloor-\lceil\frac{d}{2}\rceil} q^{-i(n-2\lfloor\frac{n}{2}\rfloor+\lceil\frac{d}{2}\rceil-1+i)} \end{aligned}$$

where $i = \lfloor\frac{n}{2}\rfloor - r$. Since $n - 2\lfloor\frac{n}{2}\rfloor \geq 0$ and $\lceil\frac{d}{2}\rceil - 1 \geq 0$, we obtain

$$\begin{aligned} A_{\rm S}(q,n,d) &< 2 + 2K_q^{-1}\theta(q)q^{(n-\lfloor\frac{n}{2}\rfloor)(\lfloor\frac{n}{2}\rfloor-\lceil\frac{d}{2}\rceil+1)} \\ &\leq 2K_q^{-1}\theta(q)q^{(\lfloor\frac{n}{2}\rfloor-r)(\lfloor\frac{n}{2}\rfloor-r+\lceil\frac{d}{2}\rceil-1)} \\ &\quad A_{\rm C}\left(q,n,r,\left\lceil\frac{d}{2}\right\rceil\right) \end{aligned} \tag{7}$$

where (7) follows from (5). □

We now compare the relation between $A_{\rm S}(q,n,d)$ and $A_{\rm C}(q,n,r,d)$ in Proposition 4 to the one determined in [11, Th. 5]. The latter only provides the following lower bound on $A_{\rm S}(q,n,d)$: $A_{\rm S}(q,n,d) \geq \frac{q^{n+1-r}+q^r-2}{q^{n+1}-1}A_{\rm C}(q,n+1,r,\lceil\frac{d}{2}\rceil+1)$. The Singleton bound on CDCs [3] indicates that $A_{\rm C}(q,n+1,r,d+1) \leq A_{\rm C}(q,n,r,d-1)$, which in turn satisfies $A_{\rm C}(q,n,r,d-1) < K_q^{-1}q^{-(n-2r-d)}A_{\rm C}(q,n,r,d)$ by (1). Hence, the lower bound on $A_{\rm S}(q,n,d)$ in [11, Th. 5] is at most $2K_q^{-1}q^{-(n-r+\lceil\frac{d}{2}\rceil-1)}A_{\rm C}(q,n,r,\lceil\frac{d}{2}\rceil)$. The ratio between our lower bound in Proposition 4 and the lower bound in [11, Th. 5] is at least $\frac{K_q}{2}q^{n-\lfloor\frac{n}{2}\rfloor+(\lfloor\frac{n}{2}\rfloor-r+1)(\lfloor\frac{n}{2}\rfloor-r+\lceil\frac{d}{2}\rceil-1)}$, and thus our lower bound in Proposition 4 is tighter than the bound in [11, Th. 5] for all cases.

The bounds in Proposition 4 help us determine the asymptotic behavior of $A_{\rm S}(q,n,d)$. We first define the rate of a subspace code $\mathcal{C} \subseteq E(q,n)$ as $\frac{\log_q|\mathcal{C}|}{\log_q|E(q,n)|}$. We note that this definition is combinatorial, and differs from the rate introduced in [3] for CDCs. The rate defined in [3] also accounts for the channel usage, but it seems appropriate for CDCs only. On the other hand, our rate depends on only the cardinality of the code, and hence is more appropriate to compare general subspace codes, since all the subspaces are treated equally regardless of their dimension. Finally, the rate defined in [3] can be derived from our rate defined here. Using the normalized parameters $r' \stackrel{\rm def}{=} \frac{r}{n}$ and $d'_{\rm S} \stackrel{\rm def}{=} \frac{d_{\rm S}}{n}$ where $d_{\rm S}$ is the minimum subspace distance of a code, the asymptotic rate of a subspace code $a_{\rm S}(d'_{\rm S}) = \limsup_{n\to\infty}\frac{\log_q A_{\rm S}(q,n,\lceil nd'_{\rm S}\rceil)}{\log_q|E(q,n)|}$ and of a CDC of given dimension $a_{\rm S}(r',d'_{\rm S}) = \limsup_{n\to\infty}\frac{\log_q A_{\rm C}\left(q,n,nr',\left\lceil n\frac{d'_{\rm S}}{2}\right\rceil\right)}{\log_q|E(q,n)|}$ can be easily determined.

Proposition 5 (Asymptotic Rate of Packing Subspace Codes in the Subspace Metric): For $0 \leq d'_{\rm S} \leq 1$, $a_{\rm S}(d'_{\rm S}) = 1 - d'_{\rm S}$. For $0 \leq r' \leq \frac{d'_{\rm S}}{2}$ or $1 - \frac{d'_{\rm S}}{2} \leq r' \leq 1$, $a_{\rm S}(r',d'_{\rm S}) = 0$; for $\frac{d'_{\rm S}}{2} \leq r \leq \frac{1}{2}$, $a_{\rm S}(r',d'_{\rm S}) = 2(1-r')(2r'-d'_{\rm S})$; for $\frac{1}{2} \leq r' \leq 1 - \frac{d'_{\rm S}}{2}$, $a_{\rm S}(r',d'_{\rm S}) = 2r'(2-2r'-d'_{\rm S})$.

Proof: First, (5) and Lemma 1 yield $a_{\rm S}(r',d'_{\rm S}) = 2(1-r')(2r'-d'_{\rm S})$ for $0 \leq r' \leq \frac{d'_{\rm S}}{2}$. Since $A_{\rm C}(q,n,r,\lceil\frac{d}{2}\rceil) = A_{\rm C}(q,n,n-r,\lceil\frac{d}{2}\rceil)$, we also obtain $a_{\rm S}(r',d'_{\rm S}) = 2(1-r')(2r'-d'_{\rm S})$ for $\frac{d'_{\rm S}}{2} \leq r \leq \frac{1}{2}$. Second, (6) for $r = \lfloor\frac{n}{2}\rfloor$ and (5) yield $a_{\rm S}(d'_{\rm S}) = a_{\rm S}(\frac{1}{2},d'_{\rm S}) = 1 - d'_{\rm S}$. □

Propositions 4 and 5 provide several important insights. First, Proposition 4 indicates that optimal CDCs with dimension being half of the block length up to rounding ($r = \lfloor \frac{n}{2} \rfloor$ and $r = n - \lfloor \frac{n}{2} \rfloor$) are optimal subspace codes up to a scalar. In this case, the optimal CDCs have a limited rate loss as opposed to optimal subspace codes with the same error correction capability. When $r < \lfloor \frac{n}{2} \rfloor$, the rate loss suffered by optimal CDCs increases with $\lfloor \frac{n}{2} \rfloor - r$. Proposition 5 indicates that using CDCs with dimension $\lceil \frac{d_S}{2} \rceil \leq r < \lfloor \frac{n}{2} \rfloor$ leads to a decrease in rate on the order of $(1-2r')(d'_S + 1 - 2r')$, where $r' = \frac{r}{n}$. Since the rate loss increases with $1 - 2r'$, using a CDC with a dimension further from $\lfloor \frac{n}{2} \rfloor$ leads to a larger rate loss.

The conclusion above can be explained from a combinatorial perspective as well. When $r = \lfloor \frac{n}{2} \rfloor$ or $r = n - \lfloor \frac{n}{2} \rfloor$, by Lemma 1, $|E(q,n)|$ is the same as $|E_r(q,n)| = \begin{bmatrix} n \\ r \end{bmatrix}$ up to scalar. Thus, it is not surprising that the optimal packings in $E(q,n)$ are the same as those in $E_r(q,n)$ up to scalar.

We also comment that the asymptotic rates in Proposition 5 for subspace codes come from Singleton bounds. The asymptotic rate $a_S(r', d'_S)$ is achieved by KK codes. The asymptotic rate $a_S(d'_S)$ is similar to that for rank metric codes [20]. This can be explained by the fact that the asymptotic rate $a_S(d'_S)$ is also achieved by KK codes when $r = \lfloor \frac{n}{2} \rfloor$, whose cardinalities are equal to those of optimal rank metric codes.

In Table I, we compare the bounds on $A_S(q,n,d)$ derived in this paper with each other and with existing bounds in the literature, for $q = 2$, $n = 10$, and d ranging from 2 to 10. We consider the lower bound in Proposition 2, its refinement in Proposition 3, and the lower bounds in [10] and [11, Th. 5] described above, and the upper bound comes from Proposition 2. Note that Proposition 4 is not included in the comparison since its purpose is to compare the cardinalities of optimal subspace codes and optimal CDCs with the same minimum subspace distance. Since bounds in Propositions 2 and 3 and [11, Th. 5] depend on cardinalities of either related CDCs or optimal CDCs, we use the cardinalities of CDCs with dimension $r = n/2 = 5$ proposed in [11] and [7] as lower bounds on $A_C(q,n,r,d)$ and the upper bound in [9] on $A_C(q,n,r,d)$ to derive the numbers in Table I. For example, the lower bound of Proposition 2 is simply given by the construction in [11] when $d = 3, 4, 5,$ and 6, and given by the construction in [7] for other values of d. Table I illustrates our lower bounds in Propositions 2 and 3 are tighter than those in [10] and [11, Th. 5]. The cardinalities of CDCs with dimension $r = n/2$ in [11] and [7], displayed as the lower bound in Proposition 2, are quite close to the lower bound in Proposition 3, supporting our conclusion that the rate loss suffered by properly designed CDCs is smaller when the dimension is close to $n/2$. Also, the lower and upper bounds in Proposition 2 depend on $\lceil \frac{d}{2} \rceil$, and hence the bounds for $d = 2l$ and $d = 2l - 1$ are the same. Finally, the tightness of the bounds improves as the minimum distance of the code increases, leading to very tight bounds for $d = n$.

C. Covering Properties of Subspace Codes With the Subspace Metric

We now consider the covering properties of subspace codes with the subspace metric. The subspace covering radius in $E(q,n)$ of a code $\mathcal{C}$ is defined as $\max_{U \in E(q,n)} d_S(U, \mathcal{C})$. We denote the minimum cardinality of a subspace code in $E(q,n)$ with subspace covering radius ρ as $K_S(q,n,\rho)$. Since $K_S(q,n,0) = |E(q,n)|$ and $K_S(q,n,n) = 1$, we assume $0 < \rho < n$ henceforth. We determine below the minimum cardinality of a code with subspace covering radius $\rho \geq \lfloor \frac{n}{2} \rfloor$.

Proposition 6: For $\lfloor \frac{n}{2} \rfloor \leq \rho < n$, $K_S(q,n,\rho) = 2$.

Proof: For all $V \in E(q,n)$, there exists $\bar{V}$ such that $V \cap \bar{V} = \{\mathbf{0}\}$ and $V + \bar{V} = \mathrm{GF}(q)^n$, and hence $d_S(V, \bar{V}) = n$. Therefore, one subspace cannot cover the whole $E(q,n)$ with radius $\rho < n$, hence $K_S(q,n,\rho) > 1$. Let $\mathcal{C} = \{\{\mathbf{0}\}, \mathrm{GF}(q)^n\}$, then for all $D \in E(q,n)$, $d_S(D, \mathcal{C}) = \min\{\dim(D), n - \dim(D)\} \leq \lfloor \frac{n}{2} \rfloor$. Thus, $\mathcal{C}$ has covering radius $\lfloor \frac{n}{2} \rfloor$ and $K_S(q,n,\rho) \leq 2$ for all $\rho \geq \lfloor \frac{n}{2} \rfloor$. □

We thus consider $0 < \rho < \lfloor \frac{n}{2} \rfloor$ henceforth. Proposition 7 can be viewed as the sphere covering bound for subspace codes with the subspace metric, as it considers how a subspace code covers each Grassmannian $E_r(q,n)$ for any $0 \leq r \leq n$.

Proposition 7 (Sphere Covering Bound for the Subspace Metric): For all q, n, and $0 < \rho < \lfloor \frac{n}{2} \rfloor$, $K_S(q,n,\rho) \geq \min \sum_{i=0}^{n} A_i$, where the minimum is taken over all integer sequences $\{A_i\}$ satisfying $0 \leq A_i \leq \begin{bmatrix} n \\ i \end{bmatrix}$ for all $0 \leq i \leq n$ and $\sum_{i=0}^{n} A_i \sum_{d=0}^{\rho} N_S(i,r,d) \geq \begin{bmatrix} n \\ r \end{bmatrix}$ for $0 \leq r \leq n$.

Proof: Let $\mathcal{C}$ be a subspace code with covering radius ρ and let A_i denote the number of subspaces with dimension i in $\mathcal{C}$. Then, $0 \leq A_i \leq \begin{bmatrix} n \\ i \end{bmatrix}$ for all $0 \leq i \leq n$. All subspaces with dimension r are covered; however, a codeword with dimension i covers exactly $\sum_{d=0}^{\rho} N_S(i,r,d)$ subspaces with dimension r, hence $\sum_{i=0}^{n} A_i \sum_{d=0}^{\rho} N_S(i,r,d) \geq \begin{bmatrix} n \\ r \end{bmatrix}$ for $0 \leq r \leq n$. □

We remark that the lower bound in Proposition 7 is based on the optimal solution to an integer linear program and hence determining this lower bound is computationally infeasible for large parameter values.

We now derive upper bounds on $K_S(q,n,\rho)$. Since $|E_{\lfloor \frac{n}{2} \rfloor}(q,n)|$ is equal to $E(q,n)$ up to a scalar, the main issue with designing covering subspace codes is to cover $E_{\lfloor \frac{n}{2} \rfloor}(q,n)$. In Proposition 8, we use subspaces in $E_r(q,n)$ in order to cover the Grassmannian $E_{r+\rho}(q,n)$ for $r \leq \lfloor \frac{n}{2} \rfloor$, i.e., $E_{\lfloor \frac{n}{2} \rfloor}(q,n)$ is covered using subspaces in $E_{\lfloor \frac{n}{2} \rfloor - \rho}(q,n)$. This choice is in fact asymptotically optimal, as we will show in Proposition 10.

The upper bound in Proposition 8 is based on the universal greedy algorithm in [14, Th. 12.2.1] to construct covering codes, which we briefly review below for subspaces. The algorithm begins by selecting as the first codeword one of the subspaces which cover the most subspaces, and then keeps adding subspaces to the code. Each new codeword is selected as to cover the most subspaces not yet covered by the code (if several subspaces cover the same number of subspaces, then the new codeword is chosen randomly). The algorithm eventually stops once all subspaces are covered. Although the cardinality of the code obtained by this algorithm is not constant, an upper bounded on its value is given in [14, Th. 12.2.1]. The bound in Proposition 8 adapts this algorithm to cover each Grassmannian $E_{r+\rho}(q,n)$ for $r \leq \lfloor \frac{n}{2} \rfloor$ by subspaces in $E_r(q,n)$. We remark that the

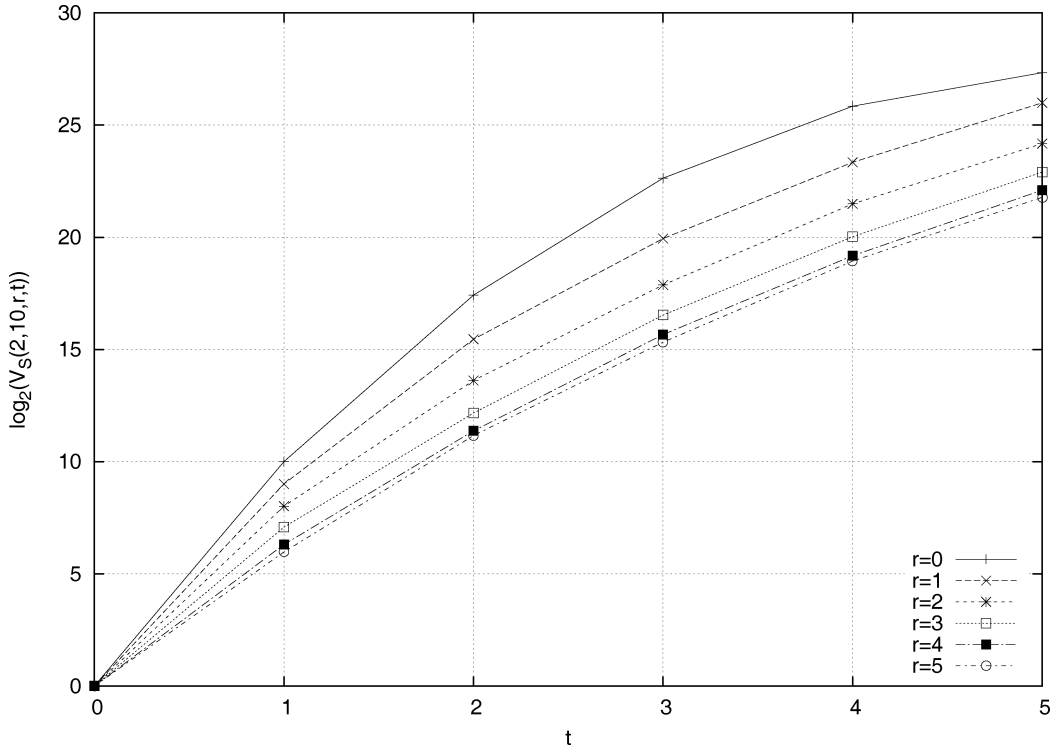

Fig. 1. Volume of a ball of subspace radius in $E(2,10)$ as a function of the dimension of its center and of its radius.

TABLE I
COMPARISON OF BOUNDS ON $A_{\mathrm{S}}(2,10,d)$ FOR d FROM 2 TO 10

	Lower bounds				Upper bound
d	Proposition 2	Proposition 3	[10]	[11, Theorem 5]	Proposition 2
2	52,494,849	59,058,177	3,181,506	3,073,032	229,755,605
3	1,167,327	1,167,967	64,047	88,163	2,616,760
4	1,167,327	1,167,329	64,047	4,650	2,616,760
5	32,841	32,843	1,986	397	50,708
6	32,841	32,841	1,986	54	50,708
7	1,025	1,025	63	12	1,260
8	1,025	1,025	63	4	1,260
9	33	33	2	2	35
10	33	33	2	2	35

bound in Proposition 8 is only semiconstructive, as it determines an algorithm to construct covering subspace codes but does not design the actual codes. We remark that the bound in Proposition 8 can be further tightened by using the bounds on the greedy algorithm derived in [21] and [22].

Proposition 8: For all q, n, $0 < \rho < \left\lfloor \frac{n}{2} \right\rfloor$, $K_{\mathrm{S}}(q,n,\rho) \leq 2 + 2\sum_{r=\rho+1}^{\left\lfloor \frac{n}{2} \right\rfloor} \lfloor k_r \rfloor$, where

$$k_r = \frac{\begin{bmatrix} n \\ r \end{bmatrix}}{\begin{bmatrix} n-r+\rho \\ \rho \end{bmatrix}} + \frac{\begin{bmatrix} n \\ r-\rho \end{bmatrix}}{\begin{bmatrix} r \\ \rho \end{bmatrix}} \ln \begin{bmatrix} n-r+\rho \\ \rho \end{bmatrix}.$$

Proof: We show that there exists a code with cardinality $2+2\sum_{r=\rho+1}^{\left\lfloor \frac{n}{2} \right\rfloor} \lfloor k_r \rfloor$ and covering radius ρ. We choose $\{\mathbf{0}\}$ to be in the code, hence all subspaces with dimension $0 \leq r \leq \rho$ are covered. For $\rho+1 \leq r \leq \left\lfloor \frac{n}{2} \right\rfloor$, let $\mathbf{A}$ be the $\begin{bmatrix} n \\ r \end{bmatrix} \times \begin{bmatrix} n \\ r-\rho \end{bmatrix}$ binary matrix whose rows represent the subspaces $U_i \in E_r(q,n)$ and whose columns represent the subspaces $V_j \in E_{r-\rho}(q,n)$, and where $a_{i,j} = 1$ if and only if $d_{\mathrm{S}}(U_i, V_j) = \rho$. Then, there are exactly $N_{\mathrm{S}}(r, r-\rho, \rho) = \begin{bmatrix} r \\ \rho \end{bmatrix}$ ones on each row and $N_{\mathrm{S}}(r-\rho, r, \rho) = \begin{bmatrix} n-r+\rho \\ \rho \end{bmatrix}$ ones on each column. By [14, Th. 12.2.1], there exists an $\begin{bmatrix} n \\ r \end{bmatrix} \times \lfloor k_r \rfloor$ submatrix of $\mathbf{A}$ with no all-zero rows. Thus, all subspaces of dimension r can be covered using $\lfloor k_r \rfloor$ codewords. Summing for all r, all subspaces with dimension $0 \leq r \leq \left\lfloor \frac{n}{2} \right\rfloor$ can be covered with $1 + \sum_{r=\rho+1}^{\left\lfloor \frac{n}{2} \right\rfloor} \lfloor k_r \rfloor$ subspaces. Similarly, it can be shown that all subspaces with dimension $\left\lfloor \frac{n}{2} \right\rfloor + 1 \leq r \leq n$ can be covered with $1 + \sum_{r=\rho+1}^{\left\lfloor \frac{n}{2} \right\rfloor} \lfloor k_r \rfloor$ subspaces. □

In Proposition 9, we design an explicit construction of a subspace covering code by combining entire Grassmannians.

Proposition 9: For all q, n, and $0 < \rho < \left\lfloor \frac{n}{2} \right\rfloor$, let

$$J_1 = \{0\} \cup \left\{ \left\lfloor \frac{n}{2} \right\rfloor - \rho - \left\lfloor \frac{\left\lfloor \frac{n}{2} \right\rfloor - \rho}{2\rho+1} \right\rfloor (2\rho+1), \ldots, \left\lfloor \frac{n}{2} \right\rfloor - 3\rho - 1, \left\lfloor \frac{n}{2} \right\rfloor - \rho \right\}$$

and $J_2 = \{i : n - i \in J_1\}$. Then, the code $\bigcup_{r \in J_1 \cup J_2} E_r(q,n)$ has subspace covering radius ρ, and hence $K_{\mathrm{S}}(q,n,\rho) \leq \sum_{r \in J_1 \cup J_2} \begin{bmatrix} n \\ r \end{bmatrix}$.

Proof: We prove that $\bigcup_{r \in J_1} E_r(q,n)$ covers all subspaces with dimension $\leq \left\lfloor \frac{n}{2} \right\rfloor$. First, all subspaces $D_0 \in E(q,n)$ with dimension $0 \leq \dim(D_0) < \left\lfloor \frac{n}{2} \right\rfloor - 2\rho - \left\lfloor \frac{\left\lfloor \frac{n}{2} \right\rfloor - \rho}{2\rho+1} \right\rfloor (2\rho+1) \leq \rho$ are covered by the subspace with dimension 0. Second, for all $D_1 \in E(q,n)$ with dimension $\left\lfloor \frac{n}{2} \right\rfloor - 2\rho - i(2\rho+1) \leq \dim(D_1) \leq \left\lfloor \frac{n}{2} \right\rfloor - \rho - i(2\rho+1)$, there exists C_1 with dimension $\left\lfloor \frac{n}{2} \right\rfloor - \rho - i(2\rho+1)$ such that $D_1 \subseteq C_1$. Thus, $d_{\mathrm{S}}(C_1, D_1) = \dim(C_1) - \dim(D_1) \leq \rho$. Similarly, for all $D_2 \in E(q,n)$ with dimension $\left\lfloor \frac{n}{2} \right\rfloor - \rho - i(2\rho+1) < \dim(D_2)$

$< \left\lfloor \frac{n}{2} \right\rfloor - 2\rho - (i-1)(2\rho+1)$, there exists C_2 with dimension $\left\lfloor \frac{n}{2} \right\rfloor - \rho - i(2\rho+1)$ such that $C_2 \subset D_2$. Thus, $d_{\mathrm{S}}(C_2, D_2) = \dim(D_2) - \dim(C_2) \leq \rho$. Therefore, $\bigcup_{r \in J_1} E_r(q,n)$ covers all subspaces with dimension $\leq \left\lfloor \frac{n}{2} \right\rfloor$. Similarly, all the subspaces with dimension $\geq n - \left\lfloor \frac{n}{2} \right\rfloor$ are covered by $\bigcup_{r \in J_2} E_r(q,n)$. □

Using the bounds derived above, we now determine the asymptotic behavior of $K_{\mathrm{S}}(q,n,\rho)$. We define $k_{\mathrm{S}}(\rho') = \liminf_{n\to\infty} \frac{\log_q K_{\mathrm{S}}(q,n,\lfloor n\rho' \rfloor)}{\log_q |E(q,n)|}$, where $\rho' = \frac{\rho}{n}$. We note that this definition of asymptotic rate is from a combinatorial perspective again.

Proposition 10 (Asymptotic Rate of Covering Subspace Codes in the Subspace Metric): For $0 \leq \rho' \leq \frac{1}{2}$, $k_{\mathrm{S}}(\rho') = 1 - 2\rho'$. For $\frac{1}{2} \leq \rho' \leq 1$, $k_{\mathrm{S}}(\rho') = 0$.

Proof: By Proposition 6, $k_{\mathrm{S}}(\rho') = 0$ for $\frac{1}{2} \leq \rho' \leq 1$. Let $\mathcal{C}$ be a KK code in $E_{\lfloor \frac{n}{2} \rfloor}(q,n)$ with minimum subspace distance $2\rho+1$ and cardinality $q^{(n-\lfloor \frac{n}{2} \rfloor)(\lfloor \frac{n}{2} \rfloor - 2\rho)}$. Then, any code $\mathcal{D} \subseteq E(q,n)$ with subspace covering radius ρ and cardinality $K_{\mathrm{S}}(q,n,\rho)$ covers all codewords in $\mathcal{C}$; however, any codeword in $\mathcal{D}$ only covers at most one codeword in $\mathcal{C}$. Hence, $K_{\mathrm{S}}(q,n,\rho) \geq q^{(n-\lfloor \frac{n}{2} \rfloor)(\lfloor \frac{n}{2} \rfloor - 2\rho)}$, which asymptotically becomes $k_{\mathrm{S}}(\rho') \geq 1 - 2\rho'$.

Also, by Proposition 8, it can be easily shown that

$$K_{\mathrm{S}}(q,n,\rho) \leq 2 + (n+1)[1 - \ln K_q + \rho(n-\rho-1)\ln q] \times K_q^{-1} q^{(n-\lfloor \frac{n}{2} \rfloor)(\lfloor \frac{n}{2} \rfloor - \rho)}$$

which asymptotically becomes $k_{\mathrm{S}}(\rho') \leq 1 - 2\rho'$. □

The proof of Proposition 10 indicates that the minimum cardinality $K_{\mathrm{S}}(q,n,\rho)$ of a covering subspace code is on the order of $q^{(n-\lfloor \frac{n}{2} \rfloor)(\lfloor \frac{n}{2} \rfloor - \rho)}$. However, a covering subspace code is easily obtained by taking the union of optimal covering CDCs (in their respective Grassmannians) for all dimensions, leading to a code with cardinality $2 + \sum_{r=\rho+1}^{n-\rho-1} K_{\mathrm{C}}(q,n,r,\lfloor \frac{\rho}{2} \rfloor)$. By [7, Prop. 11], $K_{\mathrm{C}}(q,n,r,\lfloor \frac{\rho}{2} \rfloor)$ is on the order of $q^{r(n-r)-\lfloor \frac{\rho}{2} \rfloor(n-\lfloor \frac{\rho}{2} \rfloor)}$. Hence, the code has a cardinality on the order of $q^{\lfloor \frac{n}{2} \rfloor(n-\lfloor \frac{n}{2} \rfloor) - \frac{\rho}{2}(n-\frac{\rho}{2})}$, which is greater than $q^{(n-\lfloor \frac{n}{2} \rfloor)(\lfloor \frac{n}{2} \rfloor - \rho)}$. Thus, a union of optimal covering CDCs (in their respective Grassmannians) does not result in asymptotically optimal covering subspace codes with the subspace metric.

IV. Packing and Covering Properties of Subspace Codes With the Injection Metric

A. Properties of Balls With Injection Radii

We first investigate the properties of balls with injection radii in $E(q,n)$, which will be instrumental in our study of packing and covering properties of subspace codes with the injection distance. We denote the number of subspaces with dimension s at injection distance d from a subspace with dimension r as $N_{\mathrm{I}}(r,s,d)$.

Lemma 3: $N_{\mathrm{I}}(r,s,d) = N_{\mathrm{S}}(r,s,2d-|r-s|)$. Hence, $N_{\mathrm{I}}(r,s,d) = q^{d(d+s-r)} \begin{bmatrix} r \\ d \end{bmatrix} \begin{bmatrix} n-r \\ d+s-r \end{bmatrix}$ for $r \geq s$ and $N_{\mathrm{I}}(r,s,d) = q^{d(d+r-s)} \begin{bmatrix} r \\ d+r-s \end{bmatrix} \begin{bmatrix} n-r \\ d \end{bmatrix}$ for $r \leq s$.

Proof: If $U \in E_r(q,n)$ and $V \in E_s(q,n)$, then $d_{\mathrm{I}}(U,V) = d$ if and only if $d_{\mathrm{S}}(U,V) = 2d - |r-s|$. Therefore, $N_{\mathrm{I}}(r,s,d) = N_{\mathrm{S}}(r,s,2d-|r-s|)$, and the formula for $N_{\mathrm{I}}(r,s,d)$ is easily obtained from Lemma 2. □

Lemma 3 indicates that the injection metric satisfies a strengthened triangular inequality: for any $U \in E_r(q,n)$ and $V \in E_s(q,n)$, we have $d_{\mathrm{I}}(U,V) \leq \max\{r,s\}$. We denote the volume of a ball with injection radius t around a subspace with dimension r as $V_{\mathrm{I}}(r,t) \stackrel{\mathrm{def}}{=} \sum_{d=0}^{t} \sum_{s=0}^{n} N_{\mathrm{I}}(r,s,d)$. Although the volume $V_{\mathrm{I}}(r,t)$ of a ball depends on its radius t and on the dimension r of its center, we derive below bounds on $V_{\mathrm{I}}(r,t)$ which only depend on its radius.

Proposition 11: For all q, n, r, and $t \leq \left\lfloor \frac{n}{2} \right\rfloor$, $q^{t(n-t)} \leq V_{\mathrm{I}}(r,t) < \theta(q)(2\theta(q)-1)K_q^{-2} q^{t(n-t)}$.

The proof of Proposition 11 is given in part B of the Appendix. We remark that the bounds in Proposition 11 are tight up to a scalar, which will greatly facilitate our asymptotic study of subspace codes with the injection metric. Unlike the bounds on the volume of a ball with subspace radius in Proposition 1, the lower and upper bounds in Proposition 11 do not depend on r. This illustrates a clear geometric distinction between the subspace and injection metrics.

B. Packing Properties of Subspace Codes With the Injection Metric

We are interested in packing subspace codes used with the injection metric. The maximum cardinality of a code in $E(q,n)$ with minimum injection distance d is denoted as $A_{\mathrm{I}}(q,n,d)$. Since $A_{\mathrm{I}}(q,n,1) = |E(q,n)|$, we assume $d \geq 2$ henceforth. When $d > \left\lfloor \frac{n}{2} \right\rfloor$, the maximum cardinality of a code with minimum injection distance d is determined and a code with maximum cardinality is given. For all $J \subseteq \{0,1,\ldots,n\}$, we denote the maximum cardinality of a code with minimum injection distance d and codewords having dimensions in J as $A_{\mathrm{I}}(q,n,d,J)$. For $2 \leq d \leq \left\lfloor \frac{n}{2} \right\rfloor$, we denote $Q_d = \{d, d+1, \ldots, n-d\}$. Proposition 12 relates $A_{\mathrm{I}}(q,n,d)$ to $A_{\mathrm{C}}(q,n,\left\lfloor \frac{n}{2} \right\rfloor, d)$ and shows that determining $A_{\mathrm{I}}(q,n,d,Q_d)$ is equivalent to determining $A_{\mathrm{I}}(q,n,d)$.

Proposition 12: For $d > \left\lfloor \frac{n}{2} \right\rfloor$, $A_{\mathrm{I}}(q,n,d) = 2$ and for $2 \leq d \leq \left\lfloor \frac{n}{2} \right\rfloor$, $A_{\mathrm{I}}(q,n,d) = A_{\mathrm{I}}(q,n,d,Q_d) + 2$.

Proof: Let $\mathcal{C}$ be a code in $E(q,n)$ with minimum injection distance d and let $C, D \in \mathcal{C}$. We have $\max\{\dim(C), \dim(D)\} = d_{\mathrm{I}}(C,D) + \dim(C \cap D) \geq d$, therefore there is at most one codeword with dimension less than d. Also, $\min\{\dim(C), \dim(D)\} = \dim(C+D) - d_{\mathrm{I}}(C,D) \leq n-d$, therefore there is at most one codeword with dimension greater than $n-d$. Thus, $A_{\mathrm{I}}(q,n,d) \leq 2$ for $d > \left\lfloor \frac{n}{2} \right\rfloor$ and $A_{\mathrm{I}}(q,n,d) \leq A_{\mathrm{I}}(q,n,d,Q_d) + 2$ for $d \leq \left\lfloor \frac{n}{2} \right\rfloor$. Also, adding $\{\mathbf{0}\}$ and $\mathrm{GF}(q)^n$ to a code with minimum injection distance $d \leq \left\lfloor \frac{n}{2} \right\rfloor$ and codewords of dimensions in Q_d does not decrease the minimum distance. Thus, $A_{\mathrm{I}}(q,n,d) = A_{\mathrm{I}}(q,n,d,Q_d) + 2$ for $d \leq \left\lfloor \frac{n}{2} \right\rfloor$. When $d > \left\lfloor \frac{n}{2} \right\rfloor$, $n-d \leq d$, and thus $A_{\mathrm{M}}(q,n,d) = 2$. □

Proposition 13 relates $A_{\mathrm{I}}(q,n,d)$ to $A_{\mathrm{S}}(q,n,d)$ and $A_{\mathrm{C}}(q,n,r,d)$.

Proposition 13: For all q, n, and $2 \leq d \leq \left\lfloor \frac{n}{2} \right\rfloor$, $A_{\mathrm{S}}(q,n,2d-1) \leq A_{\mathrm{I}}(q,n,d) \leq A_{\mathrm{S}}(q,n,d)$; furthermore, when $d \geq \frac{n}{3}$, $A_{\mathrm{I}}(q,n,d) \leq A_{\mathrm{S}}(q,n,4d-n,Q_d)+2$. Also, $A_{\mathrm{C}}(q,n,\left\lfloor \frac{n}{2} \right\rfloor,d) \leq A_{\mathrm{I}}(q,n,d) \leq 2+\sum_{r=d}^{n-d} A_{\mathrm{C}}(q,n,r,d)$.

Proof: A code with minimum subspace distance $2d-1$ has minimum injection distance $\geq d$ by (4) and hence $A_{\mathrm{S}}(q,n,2d-1) \leq A_{\mathrm{I}}(q,n,d)$. Similarly, a code with minimum injection distance d has minimum subspace distance $\geq d$ and hence $A_{\mathrm{I}}(q,n,d) \leq A_{\mathrm{S}}(q,n,d)$.

Let $\mathcal{C}$ be a code with minimum injection distance d whose codewords have dimensions in Q_d. For all codewords U and V, $d_{\mathrm{S}}(U,V) = 2d_{\mathrm{I}}(U,V) - |\dim(U)-\dim(V)| \geq 2d-(n-2d)$. Thus, $\mathcal{C}$ has minimum subspace distance $4d-n \geq d$ for $d \geq \frac{n}{3}$, and hence $A_{\mathrm{I}}(q,n,d,Q_d) \leq A_{\mathrm{S}}(q,n,4d-n,Q_d)$. Proposition 12 finally yields $A_{\mathrm{I}}(q,n,d) = A_{\mathrm{I}}(q,n,d,Q_d)+2 \leq A_{\mathrm{S}}(q,n,4d-n,Q_d)+2$.

Any CDC in $E_r(q,n)$ with minimum injection distance d is a subspace code with minimum injection distance d, hence $A_{\mathrm{C}}(q,n,r,d) \leq A_{\mathrm{I}}(q,n,d)$ for all r. Also, the codewords with dimension r in a subspace code with minimum injection distance d form a CDC in $E_r(q,n)$ with minimum injection distance at least d, hence $A_{\mathrm{I}}(q,n,d) = A_{\mathrm{I}}(q,n,d,Q_d)+2 \leq 2+\sum_{r=d}^{n-d} A_{\mathrm{C}}(q,n,r,d)$. □

We now derive more bounds on $A_{\mathrm{I}}(q,n,d)$. Proposition 14 is the analogue of Proposition 3 for the injection metric, and its proof is hence omitted.

Proposition 14: For all q, n, and $2 \leq d \left\lfloor \frac{n}{2} \right\rfloor$, we have $A_{\mathrm{I}}(q,n,d) \geq 2+\sum_{i=-z+1}^{z-1} A_{\mathrm{C}}(q,n,\left\lfloor \frac{n}{2} \right\rfloor - id,d)$, where $z = \left\lfloor \frac{\left\lfloor \frac{n}{2} \right\rfloor}{d} \right\rfloor$.

By extending the puncturing of subspaces introduced in [3], we finally derive below a Singleton bound for injection metric codes.

Proposition 15 (Singleton Bound for Subspace Codes in the Injection Metric): For all q,n, and $2 \leq d \leq \left\lfloor \frac{n}{2} \right\rfloor$, $A_{\mathrm{I}}(q,n,d) \leq A_{\mathrm{I}}(q,n-1,d-1) \leq \sum_{r=0}^{n-d+1} \begin{bmatrix} n-d+1 \\ r \end{bmatrix}$.

Proof: Let $W \in E_{n-1}(q,n)$. We define the puncturing $H_W(V)$ from $E(q,n)$ to $E(q,n-1)$ as follows. If $\dim(V)=0$, then $\dim(H_W(V))=0$; otherwise, if $\dim(V)=r>0$, then $H_W(V)$ is a fixed $(r-1)$-subspace of $V \cap W$. For all $U,V \in E(q,n)$, it is easily shown that $d_{\mathrm{I}}(H_W(U),H_W(V)) \geq d_{\mathrm{I}}(U,V)-1$, and hence $H_W(U) \neq H_W(V)$ if $d_{\mathrm{I}}(U,V) \geq 2$.

Therefore, if $\mathcal{C}$ is a code in $E(q,n)$ with minimum injection distance $d \geq 2$, then $\{H_W(V) : V \in \mathcal{C}\}$ is a code in $E(q,n-1)$ with minimum injection distance $\geq d-1$ and cardinality $|\mathcal{C}|$. The first inequality follows. Applying it $d-1$ times yields $A_{\mathrm{I}}(q,n,d) \leq A_{\mathrm{I}}(q,n-d+1,1) = \sum_{r=0}^{n-d+1} \begin{bmatrix} n-d+1 \\ r \end{bmatrix}$. □

We remark that although the puncturing defined in the proof of Proposition 15 depends on W, the bounds in Proposition 15 do not.

We now compare the cardinalities of optimal subspace codes and optimal CDCs with the same minimum injection distance d. We first establish the relation between $A_{\mathrm{I}}(q,n,d)$ and $A_{\mathrm{C}}(q,n,d)$ in Proposition 16.

Proposition 16 (Comparison Between Optimal Subspace Codes and CDCs in the Injection Metric): For $2 \leq d \leq r \leq \left\lfloor \frac{n}{2} \right\rfloor$

$$\begin{aligned} q^{(\left\lfloor \frac{n}{2} \right\rfloor - r)(r-d+1)} A_{\mathrm{C}}(q,n,r,d) &\leq A_{\mathrm{I}}(q,n,d) \\ &< 2K_q^{-1}\theta(q) q^{(\left\lfloor \frac{n}{2} \right\rfloor - r)(r-d+1)} A_{\mathrm{C}}(q,n,r,d). \end{aligned}$$

The proof of Proposition 16 is similar to that of Proposition 4 and is hence omitted. We also obtain another relation between $A_{\mathrm{I}}(q,n,d)$ and $A_{\mathrm{S}}(q,n,d)$.

Corollary 1: For $2 \leq d \leq \left\lfloor \frac{n}{2} \right\rfloor$

$$\begin{aligned} A_{\mathrm{S}}(q,n,2d) \leq A_{\mathrm{S}}(q,n,2d-1) &\leq A_{\mathrm{I}}(q,n,d) \\ &< 2K_q^{-1}\theta(q) A_{\mathrm{S}}(q,n,2d). \end{aligned}$$

Also, $A_{\mathrm{I}}(q,n,d) < 2K_q^{-2}\theta(q) q^{(n-\left\lfloor \frac{n}{2} \right\rfloor)(\left\lfloor \frac{n}{2} \right\rfloor - d+1)}$.

Proof: The lower bounds on $A_{\mathrm{I}}(q,n,d)$ follow Proposition 13. Furthermore, by choosing $r = \left\lfloor \frac{n}{2} \right\rfloor$ in Proposition 16, we have $A_{\mathrm{I}}(q,n,d) < 2K_q^{-1}\theta(q) A_{\mathrm{C}}(q,n,\left\lfloor \frac{n}{2} \right\rfloor,d)$. Since $A_{\mathrm{C}}(q,n,\left\lfloor \frac{n}{2} \right\rfloor,d) \leq A_{\mathrm{S}}(q,n,2d)$, we obtain $A_{\mathrm{I}}(q,n,d) < 2K_q^{-1}\theta(q) A_{\mathrm{S}}(q,n,2d)$. The last inequality follows from (5). □

Corollary 1 provides several interesting insights. First, the upper and lower bounds are all tight up to a scalar. Second, for any optimal subspace code with minimum injection distance d and cardinality $A_{\mathrm{I}}(q,n,d)$, the optimal (or nearly optimal) subspace codes with minimum subspace distance $2d$ have the same cardinality up to a scalar. Third, the last inequality in Corollary 1 implies that such nearly optimal subspace codes with minimum subspace distance $2d$ exist: KK codes in $E_{\left\lfloor \frac{n}{2} \right\rfloor}(q,n)$ are such codes.

Based on Proposition 16, we now determine the asymptotic rates of subspace codes and CDCs with the injection metric. Let us use the normalized parameters $r' = \frac{r}{n}$ defined earlier and $d'_{\mathrm{I}} \stackrel{\mathrm{def}}{=} \frac{d_{\mathrm{I}}}{n}$, where d_{I} is the minimum injection distance of a code, and define the asymptotic maximum rate $a_{\mathrm{I}}(d'_{\mathrm{I}}) = \limsup_{n\to\infty} \frac{\log_q A_{\mathrm{I}}(q,n,r,\lceil nd'_{\mathrm{I}} \rceil)}{\log_q |E(q,n)|}$ for a subspace code with the injection metric and the asymptotic rate $a_{\mathrm{I}}(r',d'_{\mathrm{I}}) = \limsup_{n\to\infty} \frac{\log_q A_{\mathrm{C}}(q,n,nr',\lceil nd'_{\mathrm{I}} \rceil)}{\log_q |E(q,n)|}$ for a CDC.

Proposition 17 (Asymptotic Rate of Packing Subspace Codes in the Injection Metric): For $\frac{1}{2} \leq d'_{\mathrm{I}} \leq 1$, $a_{\mathrm{I}}(d'_{\mathrm{I}}) = 0$; or $0 \leq d'_{\mathrm{I}} \leq \frac{1}{2}$, $a_{\mathrm{I}}(d'_{\mathrm{I}}) = 1-2d'_{\mathrm{I}}$. For $0 \leq r' \leq d'_{\mathrm{I}}$ or $1-d'_{\mathrm{I}} \leq r' \leq 1$, $a_{\mathrm{I}}(r',d'_{\mathrm{I}}) = 0$; for $d'_{\mathrm{I}} \leq r' \leq \frac{1}{2}$, $a_{\mathrm{I}}(r',d'_{\mathrm{I}}) = 4(1-r')(r'-d'_{\mathrm{I}})$; for $\frac{1}{2} \leq r' \leq 1-d'_{\mathrm{I}}$, $a_{\mathrm{I}}(r',d'_{\mathrm{I}}) = 4r'(1-r'-d'_{\mathrm{I}})$.

The proof of Proposition 17 is similar to that of Proposition 5 and hence omitted.

Propositions 16 and 17 provide several important insights on the design of subspace codes with the injection metric. First, Proposition 16 indicates that optimal CDCs with dimension being half of the block length up to rounding ($r = \left\lfloor \frac{n}{2} \right\rfloor$ and $r = n - \left\lfloor \frac{n}{2} \right\rfloor$) are optimal subspace codes with the injection metric up to a scalar. In this case, the optimal CDCs have a

TABLE II
COMPARISON OF BOUNDS ON $A_{\mathrm{I}}(2,10,d)$ FOR d FROM 2 TO 5

	Lower bounds		Upper bound
d	Proposition 13	Proposition 14	Proposition 13
2	1,167,967	1,202,145	2,616,760
3	32,843	32,843	50,708
4	1,025	1,027	1,260
5	33	35	35

limited rate loss as opposed to optimal subspace codes with the same error correction capability. When $r < \left\lfloor \frac{n}{2} \right\rfloor$, the rate loss suffered by optimal CDCs increases with $\left\lfloor \frac{n}{2} \right\rfloor - r$. Proposition 17 indicates that using CDCs with dimension $d_{\mathrm{I}} \leq r < \left\lfloor \frac{n}{2} \right\rfloor$ leads to a decrease in rate on the order of $(1-2r')(2d'_{\mathrm{I}}+1-2r')$. Similarly to the subspace metric, the rate loss for CDCs using the injection metric increases with $1-2r'$. Hence, using a CDC with a dimension further from $\left\lfloor \frac{n}{2} \right\rfloor$ leads to a high rate loss. The combinatorial explanation in Section III-B also applies in this case.

We also comment that the asymptotic rates in Proposition 17 for subspace codes come from Singleton bounds. The asymptotic rate $a_{\mathrm{I}}(r', d'_{\mathrm{I}})$ is achieved by KK codes, and the asymptotic rate $a_{\mathrm{I}}(d'_{\mathrm{I}})$ is achievable also by KK codes when $r = \left\lfloor \frac{n}{2} \right\rfloor$.

Proposition 17 also compares the difference between asymptotic rates of subspace codes with the subspace and injection metrics. Although $a_{\mathrm{S}}(d'_{\mathrm{S}})$ and $a_{\mathrm{I}}(d'_{\mathrm{I}})$ are different, the optimal subspace codes with the two metrics have similar asymptotic behavior. We note that a CDC with minimum injection distance d_{I} has minimum subspace distance $d_{\mathrm{S}} = 2d_{\mathrm{I}}$, which implies that $a_{\mathrm{S}}(r', d'_{\mathrm{S}}) = a_{\mathrm{I}}(r', d'_{\mathrm{I}})$ as long as $d'_{\mathrm{S}} = 2d'_{\mathrm{I}}$. Also, as shown above, CDCs in $E_{\left\lfloor \frac{n}{2} \right\rfloor}(q,n)$ with minimum injection distance d_{I} are both asymptotically optimal subspace codes with minimum subspace distance $d_{\mathrm{S}} = 2d_{\mathrm{I}}$ and asymptotically optimal subspace codes with minimum injection distance d_{I}. Finally, when the asymptotic rate is fixed, the relative subspace distance d'_{S} of optimal subspace codes is twice as much as the relative injection distance d'_{I}. The implication of this on the error correction capability also depends on the decoding method.

In Table II, we compare the bounds on $A_{\mathrm{I}}(q,n,d)$ derived in this paper with each other for $q=2, n=10$, and d ranging from 2 to 5 (by Proposition 12, $A_{\mathrm{I}}(2,10,d) = 2$ for $6 \leq d \leq 10$). We consider the lower bound in Proposition 13 and its refinement in Proposition 14, while the upper bound comes from Proposition 13. Note that Proposition 16 is not included in the comparison since its primary purpose is to compare the cardinalities of optimal subspace codes and optimal CDCs with the same minimum injection distance. Although some bounds rely on $A_{\mathrm{C}}(q,n,r,d)$ whose values are unknown in general, the values in Table II are obtained by using constructions in [11] and [7] as lower bounds on $A_{\mathrm{C}}(q,n,r,d)$ and the upper bound on $A_{\mathrm{C}}(q,n,r,d)$ in [9]. The cardinalities of CDCs with dimension $r = n/2$ in [11] and [7] are quite close to the lower bound in Proposition 14, again supporting our conclusion that the rate loss suffered by properly designed CDCs is smaller when the dimension is close to $n/2$. Finally, similar to the subspace distance case, the tightness of the bounds improves as the minimum distance of the code increases, leading to very tight bounds for $d = \left\lfloor \frac{n}{2} \right\rfloor$.

C. Covering Properties of Subspace Codes With the Injection Metric

We now consider the covering properties of subspace codes with the injection metric. The injection covering radius in $E(q,n)$ of $\mathcal{C}$ is defined as $\max_{U \in E(q,n)} d_{\mathrm{I}}(U, \mathcal{C})$. We denote the minimum cardinality of a subspace code with injection covering radius ρ in $E(q,n)$ as $K_{\mathrm{I}}(q,n,\rho)$. Since $K_{\mathrm{I}}(q,n,0) = |E(q,n)|$ and $K_{\mathrm{I}}(q,n,n) = 1$, we assume $0 < \rho < n$ henceforth. We first determine the minimum cardinality of a code with injection covering radius ρ when $\rho \geq \left\lfloor \frac{n}{2} \right\rfloor$.

Proposition 18: For $n - \left\lfloor \frac{n}{2} \right\rfloor \leq \rho < n$, $K_{\mathrm{I}}(q,n,\rho) = 1$. If $n = 2\left\lfloor \frac{n}{2} \right\rfloor + 1$, then $K_{\mathrm{I}}(q, 2\left\lfloor \frac{n}{2} \right\rfloor + 1, \left\lfloor \frac{n}{2} \right\rfloor) = 2$.

Proof: Let C be a subspace with dimension $\left\lfloor \frac{n}{2} \right\rfloor$. Then, for all D_1 with $\dim(D_1) \leq \dim(C)$, we have $d_{\mathrm{I}}(C, D_1) \leq \dim(C) = \left\lfloor \frac{n}{2} \right\rfloor$ by (2); similarly, for all D_2 with $\dim(D_2) \geq \dim(C) + 1$, we have $d_{\mathrm{I}}(C, D_2) \leq n - \dim(C) = n - \left\lfloor \frac{n}{2} \right\rfloor$ by (3). Thus, C covers $E(q,n)$ with radius $n - \left\lfloor \frac{n}{2} \right\rfloor$ and $K_{\mathrm{I}}(q,n,\rho) = 1$ for $n - \left\lfloor \frac{n}{2} \right\rfloor \leq \rho < n$.

If $n = 2\left\lfloor \frac{n}{2} \right\rfloor + 1$, then it is easily shown that $\{C, C^{\perp}\}$ has covering radius $\left\lfloor \frac{n}{2} \right\rfloor$, and hence $K_{\mathrm{I}}(q, 2\left\lfloor \frac{n}{2} \right\rfloor + 1, \left\lfloor \frac{n}{2} \right\rfloor) \leq 2$. However, for any $D \in E(q, 2\left\lfloor \frac{n}{2} \right\rfloor + 1)$, then either $d_{\mathrm{I}}(\{\mathbf{0}\}, D) = \dim(D) > \left\lfloor \frac{n}{2} \right\rfloor$ or $d_{\mathrm{I}}(\mathrm{GF}(q)^n, D) = n - \dim(D) > \left\lfloor \frac{n}{2} \right\rfloor$. Thus, no single subspace can cover the projective space with radius $\left\lfloor \frac{n}{2} \right\rfloor$ and $K_{\mathrm{I}}(q, 2\left\lfloor \frac{n}{2} \right\rfloor + 1, \left\lfloor \frac{n}{2} \right\rfloor) \geq 2$. □

We thus consider $0 < \rho < \left\lfloor \frac{n}{2} \right\rfloor$ henceforth. Lemma 4 relates $K_{\mathrm{I}}(q,n,\rho)$ to $K_{\mathrm{S}}(q,n,\rho)$ and $K_{\mathrm{C}}(q,n,r,\rho)$.

Lemma 4: For all q, n, and $0 < \rho < \left\lfloor \frac{n}{2} \right\rfloor$, $K_{\mathrm{S}}(q,n,2\rho) \leq K_{\mathrm{I}}(q,n,\rho) \leq K_{\mathrm{S}}(q,n,\rho)$ and $K_{\mathrm{I}}(q,n,\rho) \leq 2 + \sum_{r=\rho+1}^{n-\rho-1} K_{\mathrm{C}}(q,n,r,\rho)$.

Proof: A code with injection covering radius ρ has subspace covering radius $\leq 2\rho$, hence $K_{\mathrm{S}}(q,n,2\rho) \leq K_{\mathrm{I}}(q,n,\rho)$. Also, a code with subspace covering radius ρ has injection covering radius $\leq \rho$, hence $K_{\mathrm{I}}(q,n,\rho) \leq K_{\mathrm{S}}(q,n,\rho)$.

For $\rho + 1 \leq r \leq n - \rho - 1$, let $\mathcal{C}_r$ be a CDC in $E_r(q,n)$ with covering radius ρ and cardinality $K_{\mathrm{C}}(q,n,r,\rho)$ and let $\mathcal{C} = \bigcup_{r=\rho+1}^{n-\rho-1} \mathcal{C}_r \cup \{\{\mathbf{0}\}, \mathrm{GF}(q)^n\}$. Then, $\mathcal{C}$ is a subspace code with injection covering radius ρ and cardinality $2 + \sum_{r=\rho+1}^{n-\rho-1} K_{\mathrm{C}}(q,n,r,\rho)$. □

Proposition 19 is the analogue of Proposition 7 for the injection metric.

Proposition 19 (Sphere Covering Bound for Subspace Codes in the Injection Metric): For all q, n, and $0 < \rho < \left\lfloor \frac{n}{2} \right\rfloor$, $K_{\mathrm{I}}(q,n,\rho) \geq \min \sum_{i=0}^{n} A_i$, where the minimum is taken over all integer sequences $\{A_i\}$ satisfying $0 \leq A_i \leq \begin{bmatrix} n \\ i \end{bmatrix}$ for all $0 \leq i \leq n$ and $\sum_{i=0}^{n} A_i \sum_{d=0}^{\rho} N_{\mathrm{I}}(i,r,d) \geq \begin{bmatrix} n \\ r \end{bmatrix}$ for $0 \leq r \leq n$.

The lower bound in Proposition 19 is again based on the optimal solution to an integer linear program, and hence determining the lower bound is computationally infeasible for large parameter values.

Proposition 20 determines an upper bound on $K_{\mathrm{I}}(q,n,\rho)$, by applying the universal greedy algorithm in [14, Th. 12.2.1] to construct covering codes in the injection metric. Proposition 20

TABLE III
SUMMARY OF RESULTS

	Properties	Subspace Metric	Injection Metric
Packing	asymptotic rates	$a_{\mathrm{S}}(d'_{\mathrm{S}}) = 1 - d'_{\mathrm{S}}$	$a_{\mathrm{I}}(d'_{\mathrm{I}}) = 1 - 2d'_{\mathrm{I}}$
	optimality of CDCs with $r = n/2$	optimal up to a scalar	optimal up to a scalar
	optimal construction	optimal up to a scalar: KK codes	optimal up to a scalar: KK codes
Covering	asymptotic rates	$k_{\mathrm{S}}(\rho') = 1 - 2\rho'$	$k_{\mathrm{I}}(\rho') = (1 - 2\rho')^2$
	optimality of union of CDCs	not asymptotically optimal	asymptotically optimal
	optimal construction	asymptotically optimal semi-constructive bound: Prop. 8	asymptotically optimal semi-constructive bound: Prop. 20

is a direct application of the bound derived in [14, Th. 12.2.1] on the cardinality of a code returned by this algorithm. We remark that this bound is only semiconstructive, as it determines an algorithm to construct covering subspace codes but does not design the actual codes.

Proposition 20 (Greedy Bound for Covering Codes in the Injection Metric): For all q, n, and ρ, $K_{\mathrm{I}}(q,n,\rho) \leq \frac{|E(q,n)|}{\min_{0\leq r\leq n} V_{\mathrm{I}}(r,\rho)}\left[1+\ln\left(\max_{0\leq r\leq n} V_{\mathrm{I}}(r,\rho)\right)\right]$.

We finally determine the asymptotic behavior of $K_{\mathrm{I}}(q,n,\rho)$ by using the asymptotic rate $k_{\mathrm{I}}(\rho') = \liminf_{n\to\infty} \frac{\log_q K_{\mathrm{I}}(q,n,\lfloor n\rho'\rfloor)}{\log_q |E(q,n)|}$. According to Proposition 11, the volume of a ball with injection radius is constant up to a scalar. The consequence of this geometric result is that the greedy algorithm used to prove Proposition 20 will produce asymptotically optimal covering codes in the injection metric. However, since the volume of balls in the subspace metric does depend on the center (see Proposition 1), a direct application of the greedy algorithm for the subspace metric does not necessarily produce asymptotically optimal covering codes in the subspace metric.

Proposition 21 (Asymptotic Rate of Covering Subspace Codes in the Injection Metric): For $0 \leq \rho' \leq \frac{1}{2}$, $k_{\mathrm{I}}(\rho') = (1-2\rho')^2$. For $\frac{1}{2} \leq \rho' \leq 1$, $k_{\mathrm{I}}(\rho') = 0$.

Proof: By Proposition 18, $k_{\mathrm{I}}(\rho') = 0$ for $\frac{1}{2} \leq \rho' \leq 1$. We have

$$K_{\mathrm{I}}(q,n,\rho) \geq \frac{|E(q,n)|}{\max_{0\leq r\leq n} V_{\mathrm{I}}(r,\rho)} > \frac{K_q^2}{\theta(q)(2\theta(q)-1)} q^{\lfloor \frac{n}{2}\rfloor(n-\lfloor\frac{n}{2}\rfloor)-\rho(n-\rho)}$$

by Lemma 1 and Proposition 11. This asymptotically becomes $k_{\mathrm{I}}(\rho') \geq (1-2\rho')^2$ for $0 \leq \rho' \leq \frac{1}{2}$. Similarly, Proposition 20, Lemma 1, and Proposition 11 yield

$$K_{\mathrm{I}}(q,n,\rho) < 2K_q^{-1}\theta(q) q^{\lfloor \frac{n}{2}\rfloor(n-\lfloor\frac{n}{2}\rfloor)-\rho(n-\rho)} \left[1+\ln(\theta(q)(2\theta(q)-1)K_q^{-2}) + \rho(n-\rho)\ln q\right]$$

which asymptotically becomes $k_{\mathrm{I}}(\rho') \leq (1-2\rho')^2$ for $0 \leq \rho' \leq \frac{1}{2}$. □

The proof of Proposition 21 indicates that the minimum cardinality $K_{\mathrm{I}}(q,n,\rho)$ of a covering subspace code with the injection metric is on the order of $q^{\lfloor \frac{n}{2}\rfloor(n-\lfloor\frac{n}{2}\rfloor)-\rho(n-\rho)}$. A covering subspace code is easily obtained by taking the union of optimal covering CDCs for all constant dimensions, leading to a code with cardinality $2+\sum_{r=\rho+1}^{n-\rho-1} K_{\mathrm{C}}(q,n,r,\rho)$. By [7], the cardinality of the union is on the order of $q^{\lfloor \frac{n}{2}\rfloor(n-\lfloor\frac{n}{2}\rfloor)-\rho(n-\rho)}$. Thus, a union of optimal covering CDCs (in their respective Grassmannians) results in asymptotically optimal covering subspace codes with the injection metric.

Propositions 10 and 21 as well as their implications illustrate the differences between the subspace and injection metrics. First, the asymptotic rates of optimal covering subspace codes with the two metrics are different. Second, a union of optimal covering CDCs (in their respective Grassmannians) results in asymptotically optimal covering subspace codes with the injection metric only, not with the subspace metric. These differences can be attributed to the different behaviors of the volume of a ball with subspace and injection radius. Although $V_{\mathrm{S}}(0,t) = V_{\mathrm{I}}(0,t)$, Proposition 1 indicates that $V_{\mathrm{S}}(r,t)$ decreases with r ($r \leq \lfloor \frac{n}{2} \rfloor$), while according to Proposition 11, $V_{\mathrm{I}}(r,t)$ remains asymptotically constant. Hence, for $\lfloor \frac{n}{2} \rfloor - \rho \leq r \leq \lfloor \frac{n}{2} \rfloor$, the balls with subspace radius ρ centered at a subspace with dimension r have significantly smaller volumes than their counterparts with an injection radius. Therefore, covering the subspaces with dimension $\lfloor \frac{n}{2} \rfloor$ requires more balls with subspace radius ρ than balls with injection radius ρ, which explains the different rates for $k_{\mathrm{S}}(\rho')$ and $k_{\mathrm{I}}(\rho')$. Also, since the volume of a ball with subspace radius reaches its minimum for $r = \lfloor \frac{n}{2} \rfloor$ and $E_{\lfloor \frac{n}{2} \rfloor}(q,n)$ has the largest cardinality among all Grassmannians, using covering CDCs of dimension $\lfloor \frac{n}{2} \rfloor$ to cover $E_{\lfloor \frac{n}{2} \rfloor}(q,n)$ is not advantageous. Thus, a union of covering CDCs does not lead to an asymptotically optimal covering subspace code in the subspace metric.

V. CONCLUSION

In this paper, we derive packing and covering properties of subspace codes for the subspace and the injection metrics. We determine the asymptotic rates of packing and covering codes for both metrics, compare the performance of CDCs to that of general subspace codes, and provide constructions or semiconstructive bounds of nearly optimal codes in all four cases. These results are briefly summarized in Table III.

Despite these results, some open problems remain for subspace codes. First, our bounds on the volumes of balls derived in Lemma 1 and Propositions 1 and 11 may be tightened. Although the ratio between the upper and lower bounds is a function of the field size q which tends to 1 as q tends to infinity, it is unknown whether this ratio is the smallest that can be established. This issue also applies to the bounds on packing subspace codes in Propositions 4 and 16, where the ratios between upper and lower bounds are similar functions of q. Also, we only considered balls with radii up to $\frac{n}{2}$, as only this case was useful for our derivations; the case where the radius is above $\frac{n}{2}$ remains unexplored.

Second, the bounds on covering codes in both the subspace and the injection metrics derived in this paper are only asymptotically optimal. It remains unknown whether any of these bounds is tight up to a scalar. Third, the design of packing and covering subspace codes is an important topic for future work. This is especially the case for covering codes in the subspace metric, as no asymptotically optimal construction is known so far. Finally, the aim of this paper was to derive simple bounds on subspace codes which are good for all parameter values, especially large values. On the other hand, a wealth of *ad hoc* bounds and heuristics can be used to tighten our results for small parameter values.

APPENDIX

A. Proof of Proposition 1

Proof: When $r = 0$, we have $g(0,t) = t(n-t)$ and $V_{\mathrm{S}}(0,t) = \sum_{i=0}^{t} \begin{bmatrix} n \\ i \end{bmatrix}$ for all $t \leq \frac{n}{2}$. Hence, $V_{\mathrm{S}}(0,t) \geq \begin{bmatrix} n \\ t \end{bmatrix} \geq q^{t(n-t)}$ by (1), which proves the lower bound. Also, $V_{\mathrm{S}}(0,t) < K_q^{-1} \sum_{i=0}^{t} q^{i(n-i)} < K_q^{-1} q^{t(n-t)} \sum_{j=0}^{\infty} q^{-j^2}$ by (1), which proves the upper bound.

We now prove the bounds on $V_{\mathrm{S}}(r,t)$ for $r \geq 1$. By definition, $V_{\mathrm{S}}(r,t) = \sum_{s=0}^{n} \sum_{d=0}^{t} N_{\mathrm{S}}(r,s,d)$ is a double summation of exponential terms. The main idea of the proof is to determine the largest term in the summation: this not only gives a good lower bound, but the whole summation can also be upper bounded by that term times a constant. First, by Lemma 2, $N_{\mathrm{S}}(r,s,d) = q^{u(d-u)} \begin{bmatrix} r \\ u \end{bmatrix} \begin{bmatrix} n-r \\ d-u \end{bmatrix}$, where $u = \frac{r+d-s}{2}$ satisfies $0 \leq u \leq \min\{r,d\}$. Thus, $q^{f(u)} \leq N_{\mathrm{S}}(r,s,d) < K_q^{-2} q^{f(u)}$ by (1), where $f(u) = u(2r+3d-n-3u) + d(n-r-d)$. Hence, $\sum_{d=0}^{t} S(d) \leq V_{\mathrm{S}}(r,t) < K_q^{-2} \sum_{d=0}^{t} S(d)$, where $S(d) = \sum_{u=0}^{\min\{r,d\}} q^{f(u)}$. Since f is maximized for $u = u_0 \stackrel{\text{def}}{=} \frac{2r+3d-n}{6} \leq d$, we need to consider the following three cases.

- Case I: $0 \leq d \leq \frac{n-2r}{3}$. We have $u_0 \leq 0$ and hence f is maximized for $u = 0$: $f(0) = g(r,d) = d(n-r-d)$. Thus, $S(d) \geq q^{g(r,d)}$, and it is easy to show that $S(d) = q^{g(r,d)} \sum_{u=0}^{\min\{r,d\}} q^{-u(n-2r-3d+3u)} < \theta(q^3) q^{g(r,d)}$ since $n-2r-3d \geq 0$.
- Case II: $\frac{n-2r}{3} \leq d \leq \min\left\{\frac{n+4r}{3}, \frac{n}{2}\right\}$. We have $0 \leq u_0 \leq r$ and hence f is maximized for $u = u_0$: $f(u_0) = g(r,d) = \frac{1}{12}(n-2r)^2 + \frac{1}{4}d(2n-d)$. It is easily shown that $f(u) = f(u_0) - 3(u-u_0)^2$ for all u and hence $S(d) \geq \max\{q^{f(\lfloor u_0 \rfloor)}, q^{f(\lceil u_0 \rceil)}\} \geq q^{g(r,d)-\frac{3}{4}}$. We also obtain $S(d) = q^{g(r,d)} \sum_{u=0}^{\min\{r,d\}} q^{-3(u-u_0)^2} < 2\theta(q^3) q^{g(r,d)}$.
- Case III: $\frac{n+4r}{3} \leq d \leq \frac{n}{2}$. We have $u_0 \geq r$ and hence f is maximized for $u = r$: $f(r) = g(r,d) = (d-r)(n-d+r)$. Thus, $S(d) \geq q^{g(r,d)}$, and it is easy to show that $S(d) = q^{g(r,d)} \sum_{i=0}^{r} q^{-i(3d-4r-n+3i)} < \theta(q^3) q^{g(r,d)}$ since $3d-4r-n \geq 0$.

From the discussion above, we obtain $V_{\mathrm{S}}(r,t) \geq S(t) \geq q^{-\frac{3}{4}+g(r,t)}$ which proves the lower bound, and $V_{\mathrm{S}}(r,t) < K_q^{-2} \sum_{d=0}^{t} S(d) < 2\theta(q^3) K_q^{-2} \sum_{d=0}^{t} q^{g(r,d)}$. We now show that $R(t) = \sum_{d=0}^{t} q^{g(r,d)} < (1+q^{-1})\theta(q^{\frac{3}{4}}) q^{g(r,t)}$ by distinguishing the following three cases.

First, if $t \leq \frac{n-2r}{3}$

$$R(t) = \sum_{d=0}^{t} q^{d(n-r-d)} = q^{t(n-r-t)} \sum_{i=0}^{t} q^{-i(n-r-2t+i)} < q^{g(r,t)} \theta(q)$$

since $n-2r-2t \geq 0$.

Second, if $\frac{n-2r}{3} < t \leq \frac{n+4r}{3}$, we have $(n-2r-3d)^2 = \frac{1}{12}(n-2r)^2 + \frac{1}{4}d(2n-d) - d(n-r-d)$ and hence $\frac{1}{12}(n-2r)^2 + \frac{1}{4}d(2n-d) \geq d(n-r-d)$ for all d. We obtain

$$R(t) = \sum_{d=0}^{\lfloor \frac{n-2r}{3} \rfloor} q^{d(n-r-d)} + \sum_{d=\lfloor \frac{n-2r}{3} \rfloor + 1}^{t} q^{\frac{1}{12}(n-2r)^2 + \frac{1}{4}d(2n-d)} \leq \sum_{d=0}^{t} q^{\frac{1}{12}(n-2r)^2 + \frac{1}{4}d(2n-d)}$$

and hence $R(t) = q^{g(r,t)} \sum_{i=0}^{t} q^{-\frac{1}{4}i(2n-2t+i)} < \theta(q^{\frac{3}{4}}) q^{g(r,t)}$ since $2n-2t \geq 2t$.

Third, if $\frac{n+4r}{3} < t \leq \frac{n}{2}$, which implies $1 \leq r < \frac{n}{8}$, it can be shown that $g(r, \lfloor \frac{n+4r}{3} \rfloor) \leq g(r, \lfloor \frac{n+4r}{3} \rfloor + 1) - \frac{n-2r}{3} + 1 \leq g(r,t) - \frac{4}{3}$. Hence

$$R(t) = R\left(\left\lfloor \frac{n+4r}{3} \right\rfloor\right) + q^{g(r,t)} \sum_{j=0}^{t-\lfloor \frac{n+4r}{3} \rfloor - 1} q^{-j(n-2t+2r+j)} < q^{g(r,t)-\frac{4}{3}} \theta(q^{\frac{3}{4}}) + q^{g(r,t)} \theta(q).$$

Thus, $V_{\mathrm{S}}(r,t) < 2\theta(q^3)(1+q^{-\frac{4}{3}})\theta(q^{\frac{3}{4}}) K_q^{-2} q^{g(r,t)}$. □

B. Proof of Proposition 11

Proof: First, $V_{\mathrm{I}}(r,t) \geq N_{\mathrm{I}}(r,r,t) \geq q^{t(n-t)}$. We now prove the upper bound by determining the largest term in the double summation of $V_{\mathrm{I}}(r,t)$. Since $V_{\mathrm{I}}(r,t) = V_{\mathrm{I}}(n-r,t)$, we assume $r \leq \lfloor \frac{n}{2} \rfloor$ without loss of generality. The triangular inequality indicates that $N_{\mathrm{I}}(r,s,d) = 0$ if $s > |r-d|$ or $s > r+d$; also, by definition of the injection distance, $N_{\mathrm{I}}(r,s,d) = 0$ if $d > \max\{r,s\}$. We can hence restrict the range of parameters in the summation formula of $V_{\mathrm{I}}(r,t)$ as follows:

$$V_{\mathrm{I}}(r,t) = \sum_{d=0}^{r} \sum_{s=r-d}^{d+r} N_{\mathrm{I}}(r,s,d) + \sum_{d=r+1}^{t} \sum_{s=d}^{d+r} N_{\mathrm{I}}(r,s,d). \quad (8)$$

By Lemmas 3 and 1, we have $N_{\mathrm{I}}(r,s,d) < K_q^{-2} q^{s(n-d+r-s)-(r-d)(n-d)}$ for $s \leq r$ and $N_{\mathrm{I}}(r,s,d) < K_q^{-2} q^{s(r-s+d)+d(n-r-d)}$ for $s \geq r$, which with (8) yields

$$K_q^2 V_{\mathrm{I}}(r,t) < \sum_{d=0}^{r} \left\{ \sum_{s=r-d}^{r} q^{s(n-d+r-s)-(r-d)(n-d)} + \sum_{s=r+1}^{r+d} q^{s(r-s+d)+d(n-r-d)} \right\}$$

$$+\sum_{d=r+1}^{t}\sum_{s=d}^{d+r} q^{s(r-s+d)+d(n-r-d)}$$
$$=\sum_{d=0}^{r} q^{d(n-d)}\left\{\sum_{i=0}^{d} q^{-i(n-d-r+i)}+\sum_{j=1}^{d} q^{-j(r-d+j)}\right\}$$
$$+\sum_{d=r+1}^{t} q^{d(n-d)}\sum_{k=0}^{r} q^{-k(d-r+k)} \tag{9}$$

where we make the following changes of variables: $i=r-s$, $j=s-r$, $k=s-d$ in (9). Since $n-d-r\geq 0$, we have $\sum_{i=0}^{d} q^{-i(n-d-r+i)}<\theta(q)$. Also, $r-d\geq 0$ for $r\geq d$, and hence $\sum_{j=1}^{d} q^{-j(r-d+j)}<\theta(q)-1$; similarly, we obtain $\sum_{k=0}^{r} q^{-k(d-r+k)}<\theta(q)$. Hence, (9) leads to

$$K_q^2 V_{\mathrm{I}}(r,t)<(2\theta(q)-1)\sum_{d=0}^{r} q^{d(n-d)}+\theta(q)\sum_{d=r+1}^{t} q^{d(n-d)}$$
$$<(2\theta(q)-1)q^{t(n-t)}\sum_{l=0}^{t} q^{-l(n-2t+l)}$$
$$<(2\theta(q)-1)\theta(q)q^{t(n-t)} \tag{10}$$

where we set $l=t-d$ and use $n\geq 2r$ in (10). □

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