

# A Dispersion Theorem for Communication Networks Based on Term Sets

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**Abstract**—Traditionally, communication networks are modeled and analyzed in terms of information flows in graphs. In this paper, we introduce a new symbolic approach to communication networks, where the topology of the underlying network is contained in a set of formal terms. To any choice of coding functions we associate a measure of performance, referred to as the dispersion. Many communication problems can be recast as dispersion problems in this setup. We state and prove variants of a theorem concerning dispersion of information in communication networks which generalizes the network coding theorem. The dispersion theorem resembles the max-flow min-cut theorem for commodity networks and states that the minimal cut value can be asymptotically achieved by the use of coding functions based on a routing scheme that uses dynamic headers.

## I. INTRODUCTION

Network coding is a new and very promising technique for the transmission of data through a network [1], [2]. Unlike routing, network coding fully acknowledges the difference between digital information and ordinary commodities. It allows the intermediate nodes to combine the packets they receive and to retransmit the combined versions towards their destinations. Network coding is known to outperform routing in the case of multiple destinations [3].

A multi-user communication problem on a network is given by a network with prescribed sets of sources and of destinations, where the destinations request messages sent by the sources. The general problem of determining whether all the demands of the destinations can be satisfied at the same time (i.e. whether the instance of the problem is solvable) is crucial, and as such has been widely studied [4], [5]. This problem exhibits many pathological networks, for instance there exist communication problems which are asymptotically solvable but not solvable for any finite alphabet [6]. Different methods have been proposed to determine whether a network communication problem is solvable, including graph entropy [5] and guessing games [7].

In this paper, we represent the network communication via a set of terms, which are concepts from *logic*. A term is built on variables representing the messages sent by the sources, and on function symbols representing coding functions at the intermediate nodes. A term thus formally represents all the possible operations undergone by messages from the sources to the destinations. This novel representation has several advantages. First, the topology of the network is contained

in the term set; we can hence work without the help of the adjacency matrix of the network. Second, this graph-free framework makes computations easier to handle. Third, it is versatile and makes it for example possible to convert a dynamic multi-user network process into a static system. Fourth, it is actually more general than network coding, and hence offers not only a generalization of results in network coding, but also a reformulation in terms of flows.

We define the *min-cut* of a term set, which can be viewed as the number of degrees of freedom of the set and hence represents the information bottlenecks on the network. Conversely, to each choice of coding functions we associate a flow value, referred to as the *dispersion*, which quantifies the amount of information sent to the receivers. Also, the one-to-one dispersion is (the logarithm of) the number of outputs received for which the input can be completely determined. The main contribution of this paper is the max-flow min-cut theorem for communication networks based on term sets: the maximum dispersion and one-to-one dispersion of a term set are asymptotically equal to the value of its min-cut.

The term sets we consider may have distributed coding functions, which happens when different subterms use the same coding function. For instance, distributed coding functions occur in the term set associated to a network communication problem, where a distributed function represents the same intermediate node in terms received by different receivers. Our proof of the max-flow min-cut theorem is based on a novel protocol, referred to as dynamic routing, which works similarly to routing but uses dynamic headers to eliminate distributed functions. Clearly, this comes at a cost in bandwidth equal to the size of the header; however, this is a constant given by the term set and becomes negligible when the alphabet size increases. Dynamic routing is interesting in its own sake, as contrary to typical network coding approaches, such as random linear network coding, the manipulation of data is operated on headers only, and not on the whole packets.

The rest of the paper is organized as follows. Section II reviews some key concepts of logic and term sets and defines the analogues of flows and cuts in the new communication networks. Section III then proves the max-flow min-cut theorems for the dispersion and the one-to-one dispersion. Section V then concludes and offers some problems for future research.

## II. COMMUNICATION NETWORKS BASED ON TERM SETS

This section introduces a new type of communication networks based on term sets in logic and determines its main characteristics. We first review the basic concepts of logic, and determine an analogue of a min-cut. We then view flows as transmission of data over a given alphabet, hence determining the analogue of max-flow.

### A. Term sets

Let  $X = \{x_1, x_2, \dots, x_k\}$  be a set of *variables* and consider a set of *function symbols*  $\{f_1, f_2, \dots, f_l\}$  with respective arities (numbers of arguments)  $d_1, d_2, \dots, d_l$ . A *term* is defined to be an object obtained from applying function symbols to variables recursively. For instance, if  $k = 2$ ,  $l = 3$ , and the arities are given by  $d_1 = 1$ ,  $d_2 = d_3 = 2$ , then the following are terms:  $t_1 = f_2(f_1(x_1), x_2)$ ,  $t_2 = f_1(f_3(f_2(x_2, x_1), f_3(x_1, x_2)))$ ,  $t_3 = f_1(x_1)$ .

We say that  $u$  is a subterm of  $t$  if the term  $u$  appears in the definition of  $t$ . For instance,  $t_3$  is a subterm of  $t_1$  as  $t_1 = f_2(t_3, x_2)$ , but it is not a subterm of  $t_2$ . A proper subterm is a subterm that is not equal to the term. Furthermore,  $u$  is a *direct subterm* of  $t$  if  $t = f_j(v_1, \dots, u, \dots, v_{d_j})$ , and  $f_j$  is referred to as the *principal function* of  $t$ .

We shall consider finite term sets, typically referred to as  $\Gamma = \{t_1, t_2, \dots, t_r\}$ . We denote the set of variables that occur in terms in  $\Gamma$  as  $\Gamma_{\text{var}}$  and the collection of subterms of one or more terms in  $\Gamma$  as  $\Gamma_{\text{sub}}$ . Since each variable is a subterm and since a term is a subterm of itself we have the inclusions  $\Gamma_{\text{var}} \subseteq \Gamma_{\text{sub}}$  and  $\Gamma \subseteq \Gamma_{\text{sub}}$ .

We now define a term-cut, which can be viewed as replacing some subterms in the definition of a term by variables.

*Definition 1 (Term-cut):* A set of subterms  $s_1, s_2, \dots, s_\rho \in \Gamma_{\text{sub}}$  provides a *term-cut* of size  $\rho$  for  $\Gamma$  if all the terms can be expressed syntactically by applying function symbols to  $s_1, s_2, \dots, s_\rho$ .

A minimal term-cut for  $\Gamma$  is a term-cut with minimum size, referred to as the *min-cut* of  $\Gamma$ . The min-cut can hence be viewed as the number of degrees of freedom of the term set. Clearly the min-cut is no more than the number of variables  $k$  since  $\{x_1, x_2, \dots, x_k\}$  is a term-cut for  $\Gamma$ ; similarly, the min-cut is no more than the number of terms  $r$ .

*Example 1:* Consider the term set

$$\Gamma_1 = \{h(f(x, y), g(z, w), f(y, x)), m(g(z, w), f(y, x)), g(f(x, y), g(z, w)), f(g(z, w), f(y, x))\},$$

then the subterms  $s_1 = f(x, y)$ ,  $s_2 = g(z, w)$ , and  $s_3 = f(y, x)$  form a term-cut for  $\Gamma_1$  since we—in a purely syntactical way—can express the terms in  $\Gamma_1$  by applying function symbols to  $s_1$ ,  $s_2$ , and  $s_3$  as

$$\Gamma_1 = \{h(s_1, s_2, s_3), m(s_2, s_3), g(s_1, s_2), f(s_2, s_3)\}.$$

The concepts explained so far can be graphically explained as follows.

*Definition 2 (The graph  $G_\Gamma$ ):* For a given term set  $\Gamma$ , the directed graph  $G_\Gamma = (V, E, S, T)$  is defined to have vertex

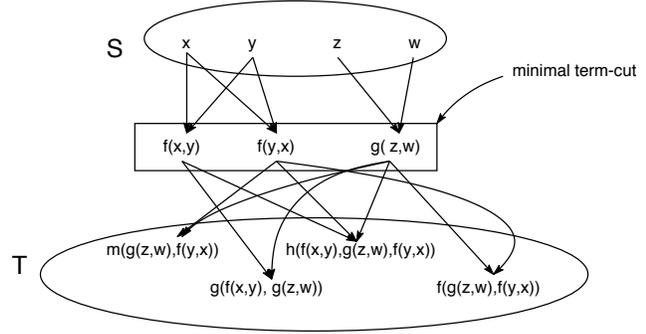


Fig. 1. The graph  $G_{\Gamma_1}$

set  $V = \Gamma_{\text{sub}}$ , edge set  $E = \{(u, v) : u \text{ directsubterm of } v\}$ , source set  $S = \Gamma_{\text{var}}$ , and target set  $T = \Gamma$ .

The graph  $G_\Gamma$  is clearly connected and acyclic. Notice that  $S \cap T$  is non-empty if  $\Gamma$  contains one or more terms that are variables.

*Example 2:* Consider the term set  $\Gamma_1$  in Example 1. The graph  $G_{\Gamma_1}$  consists of a vertex for each subterm in

$$\Gamma_{1, \text{sub}} = \{x, y, z, w, f(x, y), f(y, x), g(z, w), h(f(x, y), g(z, w), f(y, x)), m(g(z, w), f(y, x)), g(f(x, y), g(z, w)), f(g(z, w), f(y, x))\}.$$

Furthermore, each variable in  $\Gamma_{1, \text{var}} = \{x, y, z, w\}$  represents a source node and each term in  $\Gamma_1$  represents a sink (or target) node. The graph  $G_{\Gamma_1}$  is then given as in Figure 1.

Assume that  $G$  is a directed graph with source set  $S$  and target set  $T$ . We say a set  $U$  of vertices is a *vertex cut*—commonly referred to as a separating set—if the removal of  $U$  leaves no *directed* path from  $S$  to  $T$ . If  $S \cap T \neq \emptyset$  each single point in  $S \cap T$  is considered to be a path from  $S$  to  $T$ . Proposition 1 below shows that term-cuts for  $\Gamma$  are equivalent to vertex cuts in  $G_\Gamma$ .

*Proposition 1:* Assume  $\Gamma$  is a finite term set. A subset  $C \subseteq \Gamma_{\text{sub}}$  is a term-cut for  $\Gamma$  if and only if  $C$  is a vertex cut that separates  $S = \Gamma_{\text{var}}$  from  $T = \Gamma$  in the directed graph  $G_\Gamma$ . Therefore, the min-cut of  $\Gamma$  is identical to the size of the minimal cut that separates  $S$  from  $T$  in the directed graph  $G_\Gamma$ .

Due to limited space, the proof of Proposition 1 is omitted here and can be found in a complete version of this paper available online [8].

According to the directed graph version of Menger's theorem [9], there exists a family  $P$  of vertex-disjoint directed paths from  $S = \Gamma_{\text{var}}$  to  $T = \Gamma$  and a vertex cut  $C$  which consists of exactly one vertex from each path in  $P$ .

### B. Dispersion and one-to-one dispersion

So far, we have treated function symbols as abstract entities. We now study the case when these function symbols are assigned explicit values.

*Definition 3 (Model):* Let  $\Gamma = \{t_1, t_2, \dots, t_r\}$  be a term set built from the variables  $x_1, x_2, \dots, x_k$  and functions symbols  $f_1, f_2, \dots, f_l$  of arities  $d_1, d_2, \dots, d_l \in \mathbb{N}$ . Let  $A$  be a finite set with  $|A| \geq 2$ , referred to as the alphabet. A *model* for  $\Gamma$  over  $A$  is an assignment of the function symbols  $\psi_{\Gamma, A} = \{\bar{f}_1, \bar{f}_2, \dots, \bar{f}_l\}$ , where  $\bar{f}_i : A^{d_i} \rightarrow A$  for all  $1 \leq i \leq l$ .

Once all the function symbols  $f_i$  are assigned coding functions  $\bar{f}_i$ , then by composition each term  $t_j \in \Gamma$  is assigned a function  $\bar{t}_j : A^k \rightarrow A$ . In order to simplify notations, we shall write functions by the way they map a tuple  $\mathbf{a} = (a_1, a_2, \dots, a_k) \in A^k$ , and we typically write tuples in bold face. We shall abuse notations and also denote the *induced mapping* of the model as  $\psi_{\Gamma, A} : A^k \rightarrow A^r$ , defined as

$$\psi_{\Gamma, A}(\mathbf{a}) = (\bar{t}_1(\mathbf{a}), \bar{t}_2(\mathbf{a}), \dots, \bar{t}_r(\mathbf{a})).$$

Note that the definition of the induced mapping depends on the ordering of terms in  $\Gamma$ . However, our performance measures for models and induced mappings will not depend on a particular ordering. In order to simplify notations, we shall write  $\psi$  instead of  $\psi_{\Gamma, A}$  when there is no ambiguity about the term set and the alphabet.

*Example 3:* Consider  $\Gamma_1$  introduced in Example 1 and let  $A = \mathbb{F}_2$ . The model  $\psi = \{\bar{f}, \bar{g}, \bar{h}, \bar{m}\}$  given by  $\bar{f}(a_1, a_2) = a_1$ ,  $\bar{g}(a_1, a_2) = a_1 + a_2$ ,  $\bar{h}(a_1, a_2, a_3) = a_2 a_3 + 1$ , and  $\bar{m}(a_1, a_2) = a_1 a_2$  induces the mapping

$$\psi(a_1, a_2, a_3, a_4) = ((a_3 + a_4)a_2 + 1, (a_3 + a_4)a_2, a_1 + a_3 + a_4, a_3 + a_4).$$

We are especially interested in how  $\psi$  disperses its outputs, and how much information about the inputs can be obtained from the outputs. For any  $\mathbf{b} \in A^r$ , we denote the pre-image of  $\mathbf{b}$  as  $\text{pre} - \text{image}(\mathbf{b}) = \{\mathbf{a} \in A^k : \psi(\mathbf{a}) = \mathbf{b}\}$ . The image and the one-to-one image of  $\psi$  are respectively defined as

$$\begin{aligned} \text{image}(\psi) &= \{\mathbf{b} \in A^r : |\text{pre} - \text{image}(\mathbf{b})| \geq 1\}, \\ \text{one}(\psi) &= \{\mathbf{b} \in A^r : |\text{pre} - \text{image}(\mathbf{b})| = 1\}. \end{aligned}$$

We now define the analogue of the value of a flow for information transfer on networks based on term sets, which we refer to as the dispersion.

*Definition 4:* The  $\Gamma$ -dispersion and one-to-one  $\Gamma$ -dispersion of a model  $\psi$  for  $\Gamma$  over  $A$  are respectively defined as

$$\begin{aligned} \gamma(\psi) &:= \log_{|A|} |\text{image}(\psi)|, \\ \gamma_{\text{one}}(\psi) &:= \log_{|A|} |\text{one}(\psi)|. \end{aligned}$$

We remark that since  $\text{one}(\psi) \subseteq \text{image}(\psi)$ , we have  $\gamma_{\text{one}}(\psi) \leq \gamma(\psi)$  for all models  $\psi$ . For example, the model in Example 3 has  $\Gamma_1$ -dispersion of  $\log_2 6$ , while  $\psi(\mathbf{a} + (0, 0, 1, 1)) = \psi(\mathbf{a})$  for all  $\mathbf{a} \in A^4$  implies it has one-to-one  $\Gamma_1$ -dispersion  $-\infty$ .

We finally define the (one-to-one) dispersion of  $\Gamma$  over  $A$  as the maximal  $\Gamma$ -dispersion (one-to-one  $\Gamma$ -dispersion, respectively) over all models for  $\Gamma$  over  $A$ , and we denote this value by  $\gamma(\Gamma, |A|)$  (by  $\gamma_{\text{one}}(\Gamma, |A|)$ , respectively) as this quantity clearly depends on  $A$  via its cardinality only.

### III. MAX-FLOW MIN-CUT THEOREMS FOR TERM SETS

The main purpose of this section is to prove the following max-flow min-cut theorem for the dispersion and the one-to-one dispersion of term sets.

*Theorem 1 (Max-flow min-cut theorem for dispersion):*

Let  $\Gamma$  be a term set with min-cut of  $\rho$ , then for any alphabet  $A$ ,

$$\gamma_{\text{one}}(\Gamma, |A|) \leq \gamma(\Gamma, |A|) \leq \rho.$$

Conversely,

$$\lim_{|A| \rightarrow \infty} \gamma_{\text{one}}(\Gamma, |A|) = \lim_{|A| \rightarrow \infty} \gamma(\Gamma, |A|) = \rho.$$

The first part of the max-flow min-cut theorem is easily proved.

*Lemma 1:* Let  $\Gamma$  be a term set built on  $k$  variables and with min-cut of  $\rho \leq k$ . Then for all  $A$ ,  $\gamma_{\text{one}}(\Gamma, |A|) \leq \gamma(\Gamma, |A|) \leq \rho$ . Furthermore, if  $\rho < k$ , then  $\gamma_{\text{one}}(\Gamma, |A|) \leq \log_{|A|} (|A|^\rho - 1) < \rho$ .

*Proof:* Let  $C$  be a minimal term-cut for  $\Gamma$ .  $C$  can be viewed as a term set, hence let  $\psi_C$  be a model for  $C$  over  $A$ . The size of the image of its induced mapping is at most  $|A|^\rho$ . Furthermore, let  $\psi_\Gamma$  be a model for  $\Gamma$  over  $A$ . Since all terms of  $\Gamma$  can be expressed as functions of elements of  $C$ , the size of the image of  $\psi_\Gamma$  is at most that of  $\psi_C$ , hence  $|\text{image}(\psi_\Gamma)| \leq |A|^\rho$  and  $\gamma(\Gamma, |A|) \leq \rho$ .

Furthermore, if  $\rho < k$ , the average number of pre-images per element of  $\text{image}(\psi_\Gamma)$  is at least  $|A|^{k-\rho} > 1$ . Therefore, there exists an element with more than one pre-image, and  $|\text{one}(\psi_\Gamma)| \leq |A|^\rho - 1$ . ■

#### A. Diversifying term sets

We first prove the max-flow min-cut theorem for the dispersion in the specific case where each subterm has a distinct function symbol. It turns out that the process of replacing each appearance of a function symbol with distinct function symbols does not change the graph  $G_\Gamma$ . More specifically, we define the diversified term set by assigning a new function symbol to each subterm that is not a variable.

*Definition 5 (Diversified term set):* For any term set  $\Gamma$ , the *diversified term set*  $\Gamma^{\text{div}}$  is built on the same variables as  $\Gamma$  and its function symbols are obtained by replacing the principal function  $g$  of any  $u \in \Gamma_{\text{sub}} \setminus \Gamma_{\text{var}}$  by a new function symbol  $g_u$  of the same arity as  $g$ .

*Example 4:* Recall the term set  $\Gamma_1$  from Example 1, then using short-hand notation, we obtain

$$\begin{aligned} \Gamma_1^{\text{div}} &= \{h(f_1(x, y), g_1(z, w), f_2(y, x)), \\ &\quad m(g_1(z, w), f_2(y, x)), \\ &\quad g_2(f_1(x, y), g_1(z, w)), \\ &\quad f_3(g_1(z, w), f_2(y, x))\}. \end{aligned}$$

We remark that before diversification, the same function symbol may be assigned to different subterms (e.g.,  $f(x, y)$  and  $f(y, x)$  have the same principal function symbol in  $\Gamma_1$ ). However, after diversification, there cannot be such overlap, as each subterm is assigned a distinct principal function. By

definition, it is easily seen that the graph  $G_{\Gamma^{\text{div}}}$  is isomorphic to  $G_{\Gamma}$ . In particular,  $\Gamma$  and  $\Gamma^{\text{div}}$  have the same min-cut.

For diversified term sets, maximal dispersion can be achieved via *routing*, which is defined in a similar way to the case of ordinary networks. Let  $\Gamma = \{t_1, t_2, \dots, t_r\}$  be built on the variables  $\{x_1, x_2, \dots, x_k\}$  and have min-cut of  $\rho$ . According to Menger's theorem, let  $P$  be a set of  $\rho$  vertex-disjoint paths from  $\Gamma_{\text{var}}$  to  $\Gamma$  in  $G_{\Gamma}$  which, without loss, start in  $x_1, x_2, \dots, x_{\rho}$  and end in  $t_1, t_2, \dots, t_{\rho}$ , respectively.

**Definition 6 (Routing):** A distinct function symbol  $g_v$  is associated to each subterm  $v \in \Gamma_{\text{sub}}$ . If  $u_j$  is the direct subterm of  $v$  on the same path, then we let  $\bar{g}_v(a_1, a_2, \dots, a_d) = a_j$ . Otherwise, i.e. if  $v$  does not belong to any path in  $P$ , then  $\bar{g}_v(a_1, a_2, \dots, a_d) = 1$ .

Note that our definition of routing depends on the set of paths  $P$ , and hence is not unique. However, the dispersion and one-to-one dispersion of routing do not depend on the choice of  $P$ . It is straightforward to verify that using routing, all points of the form  $(a_1, a_2, \dots, a_{\rho}, 1, \dots, 1) \in A^k$  are mapped to  $(a_1, a_2, \dots, a_{\rho}, 1, \dots, 1) \in A^r$ , thus yielding a  $\Gamma$ -dispersion of  $\rho$ . Furthermore, when  $\rho = k$ , the induced mapping (restricted to the first  $\rho$  coordinates) becomes the identity on  $A^{\rho}$  and hence  $\gamma_{\text{one}}(\Gamma, |A|) = \rho$ . However, routing has one-to-one dispersion  $-\infty$  when  $k > \rho$ . In order to thwart this drawback, we define *one-to-one routing* below.

**Definition 7 (One-to-one routing):** Assume  $v$  is a subterm of the form  $g_v(u_1, u_2, \dots, u_d)$ , and denote the set of arguments equal to variables  $x_{\rho+1}, x_{\rho+2}, \dots, x_k$  as  $u_{i_1}, u_{i_2}, \dots, u_{i_m}$ . We define the coding function  $\bar{g}_v : A^d \rightarrow A$  as follows. If a path in  $P$  goes through  $v$ , denote the direct subterm of  $v$  on the same path as  $u_j$ ; then, if  $a_{i_1} = a_{i_2} = \dots = a_{i_m} = 1$ , we let  $\bar{g}_v(a_1, a_2, \dots, a_d) = a_j$ . Otherwise, let  $\bar{g}_v(a_1, a_2, \dots, a_d) = 1$ .

With one-to-one routing, it is straightforward to check that the  $(|A| - 1)^{\rho}$  points of the form  $(a_1, a_2, \dots, a_{\rho}, 1, \dots, 1) \in A^k$  with  $a_1 \neq 1, a_2 \neq 1, \dots, a_{\rho} \neq 1$  are mapped in a one-to-one fashion to  $(a_1, a_2, \dots, a_{\rho}, 1, \dots, 1) \in A^r$ , thus yielding a one-to-one  $\Gamma$ -dispersion of at least  $\rho \log_{|A|}(|A| - 1)$ .

We obtain the following max-flow min-cut result for diversified term sets.

**Proposition 2:** Assume  $\Gamma$  is a term set built on  $k$  variables and with min-cut  $\rho$ . Let  $A$  be an alphabet of size  $|A| \geq 2$ , then  $\gamma(\Gamma^{\text{div}}, |A|) = \rho$ , and it is achieved by routing. Furthermore, if  $k = \rho$ ,  $\gamma_{\text{one}}(\Gamma^{\text{div}}, |A|) = \rho$  is achieved by routing, while if  $k > \rho$ , one-to-one routing yields

$$\rho \log_{|A|}(|A| - 1) \leq \gamma_{\text{one}}(\Gamma^{\text{div}}, |A|) \leq \log_{|A|}(|A|^{\rho} - 1).$$

### B. Routing with dynamic headers

The construction of coding functions in Section III-A used the fact that each subterm  $v$  was assigned a distinct coding function. However, in general distinct subterms might be assigned the same function symbol (e.g.,  $f(x, y)$  and  $f(y, x)$ ). The proof of the general case relies on *dynamic routing*, defined below.

For  $|A| > |\Gamma_{\text{sub}}|$ , there exist two sets  $B$  and  $R$  with  $1 \leq |R| \leq |\Gamma_{\text{sub}}|$  such that  $|A| = |(\Gamma_{\text{sub}} \times B) \cup R|$  and

the union is disjoint. We shall abuse notation slightly and assume  $A = (\Gamma_{\text{sub}} \times B) \cup R$ . By construction, a tuple  $\mathbf{a} = (a_1, a_2, \dots, a_{d_j}) \in A^{d_j}$  either has an element in  $R$  or has each  $a_i = (u_i, b_i) \in \Gamma_{\text{sub}} \times B$ .

**Definition 8 (Dynamic routing):** Consider the term set  $\Gamma^{\text{div}}$  first, which contains one function symbol  $g_v$  for each subterm  $v \in \Gamma_{\text{sub}}$ . Select coding functions  $\bar{g}_v$  over  $B$  using routing, as in Definition 6. We define the functions  $\bar{f}_j(a_1, a_2, \dots, a_{d_j})$  over  $A$  as follows. If each  $a_i$  is of the form  $a_i = (u_i, b_i) \in \Gamma_{\text{sub}} \times B$ , let  $s$  denote the term  $s = f_j(u_1, u_2, \dots, u_{d_j})$ ; then if  $s \in \Gamma_{\text{sub}}$

$$\bar{f}_j(a_1, a_2, \dots, a_{d_j}) = (s, \bar{g}_s(b_1, b_2, \dots, b_{d_j})) \in \Gamma_{\text{sub}} \times B.$$

Otherwise, let  $\bar{f}_j(a_1, a_2, \dots, a_{d_j}) = r$  for some  $r \in R$ .

We can similarly define *dynamic one-to-one routing*. Remark that the headers  $u_i$  of the inputs then indicate to the coding function  $\bar{f}_j$  which subterm  $v$  it is located on, and hence which function  $\bar{g}_v$  to use. We say an input message  $a_j$  for the variable  $x_j$  is *correctly formatted* if it is of the form  $(x_j, b_j)$  where  $b_j \in B$ , and we denote the set of all corrected inputs as

$$I := \{\mathbf{a} \in (\Gamma_{\text{var}} \times B)^k : a_j = (x_j, b_j)\}.$$

Moreover, the set of correctly formatted outputs is denoted as

$$J := \{\mathbf{a} \in (\Gamma \times B)^r : a_j = (t_j, b_j)\},$$

and for all  $\mathbf{a} \in J$ , we denote the data part of  $\mathbf{a}$  as  $\mathbf{b} = (b_1, b_2, \dots, b_r) \in B^r$ . The idea behind dynamic routing is that if all inputs are correctly formatted (i.e. have the correct headers) then the coding functions  $\bar{f}_j$  mimic the behavior of the routing functions  $\bar{g}_v$ . Thus, correctly formatted messages in  $I$  are mapped in a one-to-one fashion to correctly formatted outputs in  $J$  (as long as they are mapped by the functions  $\bar{g}_v$ ), while other messages will be mapped to an "error message" in  $R$ . We obtain the following lemma.

**Lemma 2:** Let  $\psi_{\Gamma, A}$  be a dynamic routing model for  $\Gamma$  over  $A$  based on the routing model  $\phi_{\Gamma^{\text{div}}, B}$  for  $\Gamma^{\text{div}}$  over  $B$ . Then

$$\begin{aligned} \{\mathbf{a} \in J : \mathbf{b} \in \text{image}(\phi_{\Gamma^{\text{div}}, B})\} &\subseteq \text{image}(\psi_{\Gamma, A}), \\ \{\mathbf{a} \in J : \mathbf{b} \in \text{one}(\phi_{\Gamma^{\text{div}}, B})\} &\subseteq \text{one}(\psi_{\Gamma, A}). \end{aligned}$$

Lemma 2, together with Proposition 2, gives a lower bound on the dispersion and one-to-one dispersion. By choosing an appropriate alphabet size, we can prove the following quantitative version of the max-flow min-cut theorem for the dispersion.

**Theorem 2:** Let  $\Gamma$  be a term set built on  $k$  variables and with min-cut of  $\rho$ . For  $\epsilon < \rho$ , let

$$n_1 := \left( |\Gamma_{\text{sub}}| \frac{|\Gamma_{\text{sub}}|^{\rho/\epsilon}}{|\Gamma_{\text{sub}}|^{\rho/\epsilon} - |\Gamma_{\text{sub}}|} \right)^{\rho/\epsilon}.$$

Then for all  $|A| \geq n_1$ ,  $\gamma(\Gamma, |A|) \geq \rho - \epsilon$  and if  $k = \rho$ ,  $\gamma_{\text{one}}(\Gamma, |A|) \geq \rho - \epsilon$ . These are achieved by dynamic routing.

Moreover, for  $\epsilon < \frac{\rho}{1 + \log_{|\Gamma_{\text{sub}}|} 2}$ , let

$$n_2 := \left( |\Gamma_{\text{sub}}| \frac{|\Gamma_{\text{sub}}|^{\rho/\epsilon}}{|\Gamma_{\text{sub}}|^{\rho/\epsilon} - 2|\Gamma_{\text{sub}}|} \right)^{\rho/\epsilon},$$

then for all  $|A| \geq n_2$ , if  $k > \rho$ ,  $\gamma_{\text{one}}(\Gamma, |A|) \geq \rho - \epsilon$  is achieved by dynamic one-to-one routing.

*Proof:* We only prove the case involving  $n_2$ , the other being proved similarly. Suppose  $A$  is an alphabet with  $|A| \geq n_2$  and let  $\psi$  be the mapping induced by dynamic one-to-one routing for  $\Gamma$  over  $A$ . By Lemma 2 and Proposition 2,  $|\text{one}(\psi)| \geq (|B| - 1)^\rho$ . We have

$$\frac{|B| - 1}{|A|^{1-\epsilon/\rho}} \geq \frac{|A|^{\epsilon/\rho}}{|\Gamma_{\text{sub}}|} \left(1 - \frac{2|\Gamma_{\text{sub}}|}{|A|}\right) \quad (1)$$

$$\geq \frac{|A|^{\epsilon/\rho}}{|\Gamma_{\text{sub}}|} \left(1 - \frac{2|\Gamma_{\text{sub}}|}{|\Gamma_{\text{sub}}|^{\rho/\epsilon}}\right) \quad (2)$$

$$\geq 1, \quad (3)$$

where (1) follows from  $|B| \geq \frac{|A|}{|\Gamma_{\text{sub}}|} - 1$ , (2) follows from  $n_2 \geq |\Gamma_{\text{sub}}|^{\rho/\epsilon}$ , and (3) follows from the definition of  $n_2$ . Thus  $\gamma_{\text{one}}(\Gamma, |A|) \geq \log_{|A|}(|B| - 1)^\rho \geq \rho - \epsilon$ . ■

#### IV. SOLVABILITY OF TERM SETS

As Theorem 1 indicates, although it is not always possible to reach the min-cut for any fixed finite alphabet, this can be achieved asymptotically. The class of term sets  $\Gamma$  can then naturally be divided into two disjoint classes whether there exist coding functions that achieve perfect dispersion equal to the min-cut. If  $\Gamma$  has perfect dispersion, we also say  $\Gamma$  is *solvable* and a *solution* is a model with dispersion equal to the min-cut.

Solvable term sets are easily found (trivially, any term set consisting of variables). We now exhibit a non-solvable term set:

$$\Gamma = \{f(x, y), f(x, z), f(w, y), f(w, z)\}.$$

$\Gamma$  has min-cut 4 ( $\Gamma_{\text{var}}$  is a term-cut of size 4; conversely  $(x, f(x, y))$ ,  $(z, f(x, z))$ ,  $(y, f(w, y))$ , and  $(w, f(w, z))$  are 4 vertex-disjoint paths from  $\Gamma_{\text{var}}$  to  $\Gamma$ ). By Proposition 2, the dispersion of the diversified term set is  $\gamma_{\text{one}}(\Gamma^{\text{div}}, |A|) = \gamma(\Gamma^{\text{div}}, |A|) = 4$  for all  $|A| \geq 2$ , which can be achieved by routing. In other words, all demands can be satisfied independently. However, Proposition 3 shows that the whole problem does not have any solution.

*Proposition 3:* For any alphabet  $A$ ,  $\gamma(\Gamma, |A|) < 4$ .

*Proof:* Let  $\bar{f}$  be an assignment for the coding function  $f$  and  $\psi$  be the corresponding induced mapping. Consider the subset of inputs  $B = \{\mathbf{a} \in A^4 : a_4 = a_1 \in A\}$  with size  $|B| = |A|^3$ . Since  $\psi(a_1, a_2, a_3, a_1) = (\bar{f}(a_1, a_2), \bar{f}(a_1, a_3), \bar{f}(a_1, a_2), \bar{f}(a_1, a_3))$ , the image of  $B$  has size at most  $|A|^2$  and hence  $|\text{image}(\psi)| < |A|^4$ . ■

We would like to illustrate the difference between solvability and reaching the min-cut asymptotically by considering  $A = \mathbb{F}_2$ . In this case, there are  $2^{2^2} = 16$  choices for the coding function  $\bar{f}$ . However, it can be easily shown that any function is equivalent to one of the following four functions:  $\bar{f}_0(a_1, a_2) = 0$ ,  $\bar{f}_1(a_1, a_2) = a_1$ ,  $\bar{f}_2(a_1, a_2) = a_1 + a_2$ , and  $\bar{f}_3(a_1, a_2) = a_1 a_2$ . We easily obtain that  $\bar{f}_0$ ,  $\bar{f}_1$ , and  $\bar{f}_2$  have dispersion 0, 2, and 3, respectively; since they are linear functions, they all have one-to-one dispersion equal to  $-\infty$ . On the other hand, the image of  $\bar{f}_3$  consists of 10 elements: the

9 elements of its one-to-one image together with the all-zero vector. Therefore,  $\bar{f}_3$  has dispersion  $\gamma(\bar{f}_3) = \log_2 10 \equiv 3.32$  and one-to-one dispersion  $\gamma_{\text{one}}(\bar{f}_3) = \log_2 9 \equiv 3.17$ . Thus, there is a clear gap between the maximal dispersion (and one-to-one dispersion) achievable for a given alphabet and the min-cut, which is achievable asymptotically.

#### V. CONCLUSION

There is an extensive literature for dealing with the logistics and scheduling in traditional commodity networks. The theories are very diverse ranging from linear programming, algorithms for transport of “discrete” goods, game theory, traffic flow theory, network exchange theory, economic network theory, packet switching and queuing theory. Surprisingly, there have been very few attempts to develop similar theories that cover transport of digital information in communication networks. We believe that there is a strong need to develop network coding theories for transport of digital information analogous to theories of transport of ordinary commodities.

In this paper, we considered communication networks based on term sets. These networks are shown to generalize typical views of network coding, and offer an interpretation as flows of information. In particular, the max-flow min-cut theorem indicates that the maximum amount of information that can be transmitted through a network can be viewed as the min-cut of the term set. Also, the use of dynamic routing allowed us to virtually eliminate distributed coding functions.

This topic is studied further in [8], where different measures of information received by the destinations, namely the Rényi entropy [10], are considered. In particular, the max-flow min-cut theorem for the dispersion can be extended to the Rényi entropy for a limited range of orders. On the other hand, the max-flow min-cut theorem for the one-to-one dispersion is completely independent from the result for the Rényi entropy.

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